Relaxation at the Angle of Repose

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We have investigated the distribution of avalanches which occur when the slope of a sandpile exceeds the metastable slope given by the angle of repose. In contradiction to recent models by Bak, Tang, and Wiesenfeld for the dynamics of granular systems, we do not observe power-law distributions. We also find that the slope of a sandpile decays as log(t) when vibrations are introduced. We propose a simple model for this effect.

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Sandpiles have often been used as a metaphor for other phenomena in physics. As children we have played with sand and it is only natural to believe that we can intuit the processes underlying the shape and dynamics of these structures. Thus de Gennes[1] has compared the motion of vortices in superconductors to avalanches in sandpiles, and Souletie[2] has argued that spin-glasses and sandpiles behave in a similar manner. Dry, noncohesive, granular material like sand can flow, resembling a liquid, but also, like a solid, can sustain under the influence of gravity a finite "angle of repose," θ_r. This is the angle between the horizontal and the free surface of the sandpile after a land slide has restored the pile to a metastable equilibrium slope. As early as 1773 the connection between θ_r and the static internal friction of a given material was developed by Coulomb.[3] The analogies mentioned above were made because it is tempting to think of a phase transition occurring at θ_r: For slopes such that $\theta < \theta_r$ no flow of sand can occur and the pile appears to be a solid whereas for $\theta > \theta_r$ the top layers of sand flow freely downhill. Most recently Bak, Tang, and Wiesenfeld[4] introduced the idea of "self-organized criticality" in terms of a model of how they expected a sandpile to behave. Clearly their ideas may have much wider applicability than to just sandpiles, but it is this analogy which makes their model so intuitively appealing. Their idea rests on the assumption that θ_r is a critical angle. If the angle θ of the free surface is increased continuously (e.g., by adding more material to the top or by tilting the base of the pile), the pile will organize itself such that its average slope will be the angle of repose θ_r by unloading excess material through avalanches. Bak, Tang, and Wiesenfeld predict a self-organized state at θ_r characterized by long-range spatial and temporal correlations and giving rise to a typical "$1/f$" power spectrum of the fluctuations around the steady-state particle flow. They further suggest an analogy between the nonequilibrium behavior of sandpiles and traditional critical phenomena and find a power-law dependence for the relaxation from a supercritical state $\theta > \theta_r$ back to the critical state.[5]

Motivated by the above conjectures we have investigated the nature of particle flow along the free surface in a model system for granular materials: "sandpiles" consisting of spherical glass beads or rough aluminum-oxide particles. We report here on the power spectrum for fluctuations as well as on relaxation properties.

We have performed the experiment in two basic configurations [see Fig. 1(a)]. The average slope of the free surface was varied either by turning a semicircular drum (i.e., tilting the free surface of the pile) partially filled with granular material, or by randomly adding particles to the top surface of a pile contained in a box with one open side. The semicircular drum of width 8 cm and radius 5 cm had an angular velocity $\Omega = 1.3°/\text{min}$. Several box geometries were used having a width of ap-

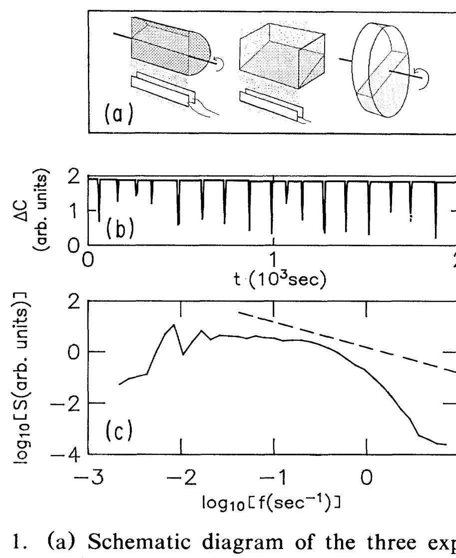

FIG. 1. (a) Schematic diagram of the three experimental configurations (left to right): A rotating semicircular drum, an open box with sand added from the top, and a closed cylindrical drum. In the first two the sand falls through a pair of capacitor plates. (b) The time trace of a typical sequence of avalanches in a rotating drum ($\Omega = 1.3°/\text{min}$) with spherical glass beads. ΔC is the change in capacitance due to the flow of the beads through the capacitor. (c) The power spectrum, $S(f)$, for the above time trace. Dashed line shows a $1/f$ power spectrum for comparison.

proximately 5 cm and aspect ratios (width/length) ranging from $\frac{1}{3}$ to 3. In the drum we used spherical glass beads of average diameter 0.54 ± 0.08 mm. For the open box we used smaller beads, 0.07 ± 0.01-mm average diameter, to keep the momentum transfer as low as possible when the particles hit the surface of the pile. The results in both configurations were insensitive to the particle size although for the very small particles the experiments were performed in a drybox since special care had to be taken to prevent cohesion due to moisture. We also performed the experiment with rough aluminum-oxide particles of diameter 0.5 ± 0.1 mm.

The flow of particles over the open edge of the drum or box was detected by a parallel-plate capacitor through which the particles flowed. The corresponding time sequence of capacitance changes, ΔC, was detected by a bridge circuit and fed into a spectrum analyzer. Capacitance changes directly correspond to changes in the flow rate of particles. The resolution was sufficient to detect even the passage of only a few particles at a time.

We show, in Fig. 1(b), a typical time trace and, in Fig. 1(c), the corresponding power spectrum obtained with the rotating drum using the spherical glass beads. Similar results were obtained with the open box and for the case of rough aluminum-oxide particles. Each inverted spike in Fig. 1(b) corresponds to a major avalanche, in almost all cases spanning the whole free surface of the system. The avalanches start at an angle $\theta_m \equiv \theta_r + \delta$ (where δ is typically a few degrees) and return the pile to its metastable state at θ_r. The distributions for the Δt interval between avalanches and their width τ are sharply peaked near their average values as shown in Fig. 2. The nearly uniform spacing of avalanches, $\langle \Delta t \rangle = \langle \Delta\theta \rangle / \Omega$, corresponds to the broadened peak at $f = 1/\langle \Delta t \rangle$ in the power spectrum. The roll off at high frequencies above $f = 1/\langle \tau \rangle$ is due to the finite width τ of the individual avalanches. Its average power-law decay, close to f^{-3}, is explained by variations between rectangular and triangular pulse shapes of individual events. In the relevant frequency range, between $1/\langle \Delta t \rangle$ and $1/\langle \tau \rangle$, the spectra appear frequency independent. This finding is in conflict with Bak, Tang, and Wiesenfeld who predicted power-law behavior, $f^{-\theta}$, with $\theta \cong 1$. Such behavior would have followed from a power-law distribution of widths τ implying events over a wide range of time scales. Instead we find that the power spectrum is indicative of a linear superposition of global, system-spanning avalanches with a narrowly peaked distribution of time scales.

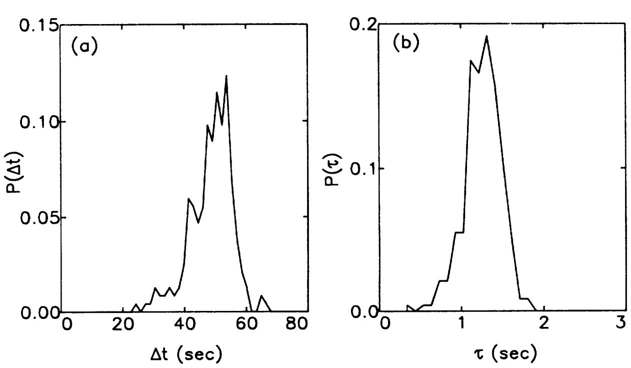

FIG. 2. (a) The distribution of intervals between avalanches, $P(\Delta t)$, and (b) the distribution of avalanche durations, $P(\tau)$, for a run using glass beads in the rotating drum ($\Omega = 3.2°$/min).

This situation is a direct consequence of the existence of a nonzero value of δ. The phenomenon was first explained as dilatation by Reynolds[6] in 1885. For particles to slide on the surface, θ_r (which corresponds to the metastable configuration after an avalanche) must be exceeded by an additional amount δ to allow for clearance with the profile of the underlying layer.[7] As long as effects from the retaining walls of the container can be neglected we find $\langle \theta_r \rangle = 26°$ (39°) and $\langle \delta \rangle = 2.6°$ (5°) for glass beads (aluminum-oxide particles), independent of container or grain size. These values for $\langle \delta \rangle$ agree with calculations by Bagnold[7] who also predicted the periodic occurrence of avalanches in the steady state. We believe that the observation of a wide spread of values for Δt by Evesque and Rajchenbach[8] is due to wall effects in a comparatively narrow system.

Since θ_m is inherently unstable, the hysteresis, δ, is easily reduced to zero by the application of mechanical vibrations to the system. By adjusting the intensity of the vibrations one can span the transition from solidlike to liquidlike behavior of the granular material in a very controlled way. We induced vibrations by a speaker mechanically coupled to the system. This configuration allowed us in a convenient way to address whether $1/f$ fluctuations in the steady-state particle flow for $\Omega = \text{const}$ could be recovered if $\delta \to 0$. It also allowed us to investigate, for $\Omega = 0$, the time dependence of $\theta(t)$ as the average angle of the free surface relaxes under the influence of the vibrations.

Figure 3 shows the variation of the power spectrum as the vibration intensity in the drum is increased. The spectra show that increasing intensity leads to a broader distribution of time scales, yet no universal power-law behavior emerges. Instead the fall off in the curve broadens smoothly as the intensity is increased and at the highest vibration intensity the spectrum can be approximated by $f^{-0.8}$ over a limited range. If a finite value of δ were the main reason for the absence of universal power-law dependence, that dependence should have emerged for vibration intensities even smaller than that corresponding to the top curve in Fig. 3. Clearly the introduction of noise does not bring the system closer to a critical state. The transition that occurs appears to be more similar to a first-order transition than to a critical point.

We have studied the nature of the particle flow by in-

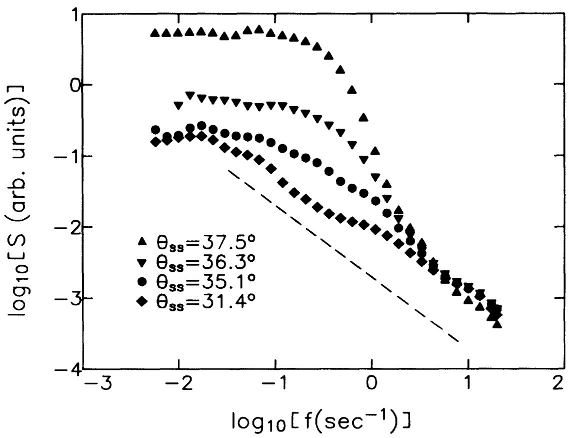

FIG. 3. The power spectra for avalanches of aluminum-oxide particles in a rotating drum ($\Omega = 1.3°/\mathrm{min}$) for different vibration intensities (increasing from top to bottom) parametrized by the steady-state angle, θ_{ss}. Dashed line shows a $1/f$ power spectrum for comparison. Similar results were found for spherical beads.

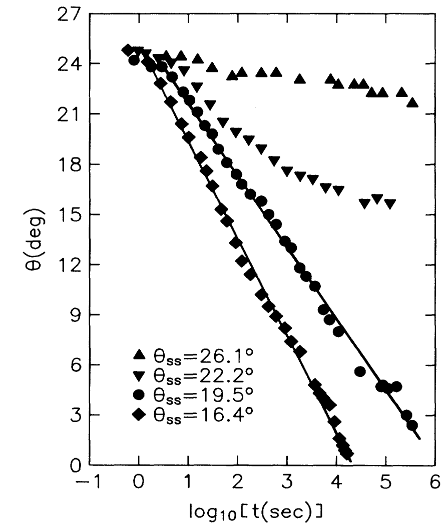

FIG. 4. The relaxation of θ in a stationary drum with glass beads. Vibration intensities increase from top to bottom. Straight lines indicate $\log(t)$ behavior.

vestigating the dependence of the average slope on vibration intensity and rotation speed Ω. The data were taken both in the semicircular drum and in a fully circular drum with no opening for particles to leave [Fig. 1(a)]. This second configuration had radius 15 cm and width 10 cm and allowed continuous rotation without refilling. The steady state of the system at constant speed Ω and constant nonzero vibration intensity is characterized by an angle θ_{ss}. Switching off the vibrations thus is a way to prepare the system in a supercritical state, $\theta > \theta_{ss}$, and one can then observe the relaxation to the steady state as the vibrations are turned back on. Such relaxation is shown in Fig. 4 for the case $\Omega = 0$ (stationary drum) and initial condition $\theta(t=0) = \theta_r$. The model of self-organized criticality predicts[4,9] a power-law dependence on t while our data cannot be fitted by a power law with reasonable parameters over any wide interval of time, even for small intensity. Instead the data for high vibration intensity are consistent with a $\log(t)$ dependence over many decades. For smaller intensity the curves follow a $\log(t)$ behavior over several decades but turn over to a slower decay rate at very large times.

To make such dependence plausible, consider a simple model based on the notion that the vibration intensity plays the role of an effective temperature. In analogy with electrical conduction we relate the particle flow j to an applied field E by a conductivity σ whose T dependence is governed by an effective rate of escape from random trapping sites. For the rotating drum we replace j by $d\theta/dt$ and E by θ. The motion of particles is impeded by the barriers posed by neighboring beads and the corresponding random potential will be a complicated function of the local configurations. Here, however, we are interested in the average effective barrier height U as the angle θ is varied. Expanding to first order around the experimental starting value $\theta(t=0) = \theta_r$ we have $U \cong U_0 + U_1(\theta_r - \theta)$. Since we know that spontaneous flow sets in at θ_m we require $U(\theta_m) = 0$ which gives $\delta \equiv \theta_m - \theta_r = U_0/U_1$. Assuming now an effective temperature T_{eff} due to the mechanical vibrations, we obtain a rate of escape over the barrier exponentially dependent on the ratio U/kT_{eff}. This leads to

$$\frac{d\theta}{dt} = -A\theta e^{\beta(\theta - \theta_r)},$$

where $A \equiv A_0 \exp(-U_0/kT_{\mathrm{eff}})$ and $\beta \equiv U_1/kT_{\mathrm{eff}}$ are independent of θ. The equation can be solved analytically for $t(\theta)$ in terms of the exponential integral function[10] $E_1(\beta\theta)$. For $\beta\theta \gg 1$ and, up to logarithmic corrections in θ/θ_r, the solution is well approximated by $\theta = \theta_r - (1/\beta)\ln(\beta A \theta_r t + 1)$. This reproduces the logarithmic dependence on t seen in Fig. 4 for times larger than $t_0 \equiv 1/\beta A \theta_r$ and also gives a good fit for shorter times.

For small vibration intensities the assumption that mechanical vibrations mimic an effective temperature fails when kT_{eff} becomes less than U. In contrast to thermal fluctuations we expect the mechanical energy distribution to be cut off sharply above a finite value corresponding to the maximum vibration intensity. Particles therefore get stuck when $\beta(\theta_r - \theta) < c$, where c is some constant of order unity, and the logarithmic decay seen in Fig. 4 flattens out. Also at very long times it becomes difficult to determine the angle θ precisely since the profile of the free surface starts to deviate from a straight line. θ in Fig. 4 is therefore the average angle, measured along the center portion of the free surface to exclude boundary effects. At long times, when most particle flow downhill has ceased, convective motion in the bulk may become important.[11]

In conclusion, we find that the simple, intuitive picture of θ_r as the critical angle of the system has to be modified and that the direct analogy between the dynamics of sandpiles and physical systems exhibiting a critical point, suggested by Bak, Tang, and Wiesenfeld, is not well founded. Perhaps other systems can be found which would show such critical behavior. Under steady-state conditions the angle θ oscillates between θ_r and θ_m. However, the observation that $\theta(t) \propto T_{\text{eff}} \log(t)$ when there is a finite vibration intensity suggests that the relaxation of the free surface in sandpiles might be related to phenomena such as flux creep in superconductors[12] and slow relaxation processes in glasses. In addition, further investigation of the question of whether the models introduced by Bak, Tang, and Wiesenfeld have universal properties is currently under way.[13] Finally, an understanding of the dynamics of particle flow in sandpiles is important not only for its use as a metaphor to other physical systems but also because granular materials are widely used in industrial processes. Although much is known about the time-independent bulk flow properties of granular materials very little is known about the instabilities and fluctuations that occur in these flow patterns. Dynamical effects of this kind are suspected of being responsible for the collapse of grain silos.[14] Only recently have attempts been made[14,15] to investigate such problems. Studies of what happens at the surface of sandpiles may be one way of making accessible the underlying physics of granular material.

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