Stochastic Resonance in Bistable Systems

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The relaxation properties of a stochastic bistable system perturbed by a periodic low-frequency forcing term is investigated by means of analog simulation. The so-called stochastic resonance phenomenon is revealed under diverse experimental conditions. Its dependence on the parameters of both the periodic perturbation and the forced system is explained theoretically.

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It has been noticed[1] that the output signal from a stochastic bistable system may be modulated in time by applying an external periodic perturbation. Such an effect is apparent even when the perturbation is weak enough not to appreciably affect the noise-induced switch process. The interplay of intrinsic noise and periodic driving mechanism results in a sharp enhancement of the signal power spectrum within a narrow range about the forcing frequency. This observation was explained by Benzi *et al.*[2] by relating the forcing frequency with the switch rate (Kramers rate) of the unperturbed system between two adjacent stable states. To distinguish this phenomenon from the well-known[3] dynamical resonance one speaks of *stochastic resonance* (SR).[1]

Recently, McNamara, Wiesenfeld, and Roy[4] pointed out that a more suggestive signature for SR is given by the signal-to-noise ratio (SNR). They measured the SNR for an optical bistable device (a bidirectional ring laser), showing that it increases dramatically with the noise intensity, peaks at switch-energy-to-noise-intensity ratio equal to two, and fades away smoothly for stronger noise intensities.

The main focus of this Letter is on two important features of SR which are essential[4] for a complete understanding of the phenomenon, namely, its dependence on the frequency and intensity of the periodic modulation and its persistence under a variety of physical circumstances. Our investigation is based on analog simulation techniques, the most sophisticated version[5] of which proved to be more accurate than numerical simulation and also allows one to reproduce more realistic experimental conditions than in previous studies.[1,6] The results reported below refer to the simulation of a one-dimensional quartic double-well potential $V(x) = -ax^2/2 + bx^4/4$, subject to both *fluctuation and dissipation.* The periodic driving mechanism is described by a sinusoidal forcing term (with arbitrary phase) $A(t) = A\cos(\Omega t)$. The corresponding stochastic differential equation reads

$$\ddot{x} = -\gamma\dot{x} + ax - bx^3 + A\cos(\Omega t) + f(t)\,, \qquad (1)$$

where a, b, and γ are positive and the Gaussian, zero-mean valued noise $f(t)$ is assumed to be δ-correlated, i.e., $\langle f(t)f(0)\rangle = 2\gamma D\,\delta(t)$.

In practice, $f(t)$ is simulated by means of a correlated fluctuating voltage, the correlation time of which (50 μs) is negligible compared to the characteristic time scales of the analog circuit (> 1 ms).[7] In particular, the deterministic resonance frequency about the potential minima is $\nu_0 = \omega_0/2\pi = 1650$ Hz. For the setup of our simulation apparatus we refer the reader to a previous publication.[5] The output signal analysis was performed by means of a Data 6000 waveform analyzer. This instrument digitizes (with 14-bit resolution) and stores (with a record-buffer length of up to 4 kilobytes) analog signals which are then processed in real time to compute the statistical quantities of interest. In the present case, the parameters γ, D, A, and Ω may be varied within a very wide range of values. This allowed us to verify the occurrence of SR in both the overdamped and underdamped regime for any value of the forcing frequency $\nu_\Omega = \Omega/2\pi$ smaller than the Kramers rate for the unperturbed (bistable[8]) system, $2\mu_K$.

A way of characterizing the notion of SR proposed by Benzi and co-workers[1,2] is to look at a discrete stochastic process defined indirectly by Eq. (1). Let $T(n)$ denote the first-crossing time of the nth sampling record of the output signal $x(t)$. $T(n)$ can be measured by means of the Data 6000 (after arming its internal trigger suitably) and stored in a record buffer for further processing. The first-crossing time thus determined corresponds to measuring the switch time of $x(t)$ between its stable values $\pm x_m$. (In our analog circuit x_m has been chosen equal to 7.3 V.) In Fig. 1 we display the distribution of $T(n)$, $N(T)$, taken over a set of 5000 records for several values of Ω with γ and D fixed. At $A=0$ we recover the usual distribution of the first-passage times, $T_K^{-1} \times \exp(-T/2T_K)$, where T_K is the reciprocal of the Kramers rate. For the parameter values of Fig. 1(a) $T_K = 14.8 \pm 0.1$ ms, in good agreement with the relevant theoretical predictions.[8] In the presence of the deterministic forcing term $A(t)$, instead, the number of crossings peaks at $T \approx \pi/\Omega$. This implies that the process $x(t)$ switches almost periodically between its stable minima with frequency ν_Ω, as first observed in Ref. 1.

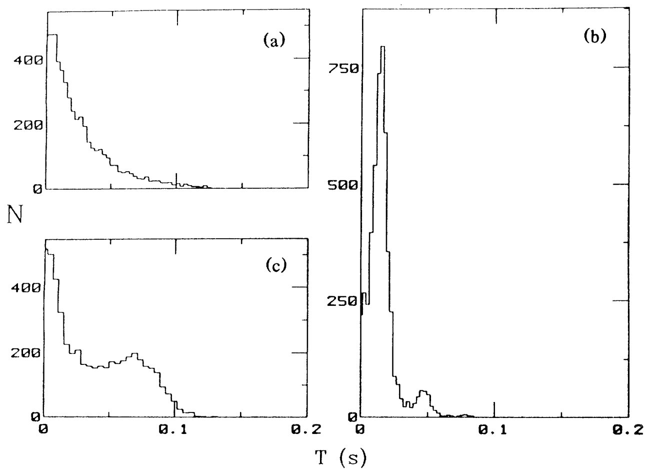

FIG. 1. First crossing-time distribution $N(T)$ for (a) $A=0$, (b) $Ax_m=0.5\Delta V$ and $\nu_\Omega=30$ Hz, and (c) $Ax_m=0.5\Delta V$ and $\nu_\Omega=6$ Hz. Other parameter values; $x_m=7.3$ V, $\nu_0=1650$ Hz, $\Delta V/D=3$, and $\gamma/\omega_0=0.64$. The estimated error is less than 5%.

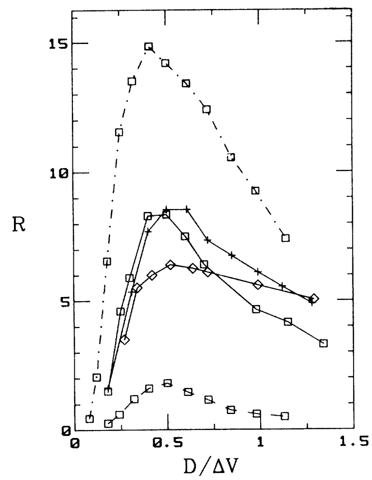

FIG. 2. The SNR vs D at several values of Ax_m (dashed line, $0.22\Delta V$; solid line, $0.5\Delta V$; dotted-dashed line, $0.66\Delta V$) and ν_Ω (squares, 15 Hz; crosses, 30 Hz; lozenges, 500 Hz). The other parameter values are as in Fig. 1. The relevant averages have been taken over 5000 digitized spectra (Ref. 5).

The peak of $N(T)$ reaches its maximum for $\nu_\Omega \approx 30$ Hz. For lower forcing frequencies the SR peak smoothes out, whereas in the opposite limit it merges into the exponentially decaying branch of $N(T)$ about $T=0$. Furthermore, at high frequencies [Fig. 1(b)] side peaks located about the odd multiples of π/Ω show up in analogy with the subharmonics of dynamical resonance in nonlinear systems.

A different characterization of SR has been suggested by the authors of Ref. 4. Let us consider the continuous process $x(t)$ in (1). The background of its power spectrum $S(\nu)$ increases with D. A δ-like spike corresponding to the appearance of SR is located in ν_Ω and its intensity grows with D up to a maximum for about $\mu_K \approx \nu_\Omega$ [like the peak of $N(T)$] and then plunges into the background again for larger D. The SNR is the best candidate to quantify the effect of interest. However, the experimental determination of the SR peak $S(\nu_\Omega)$ requires some caution because its height depends on the analyzer sampling time. Throughout the present Letter all of our measurements of $S(\nu_\Omega)$ are referred to a conventional bandwidth $\Delta\nu_{\text{expt}}=5$ Hz. No such difficulty arises for the remainder of $S(\nu)$ because of the Wiener-Khintechine theorem.[5] The background of the power spectrum about ν_Ω, $B(\nu_\Omega)$, is determined by linear interpolation of the discretized $S(\nu)$ after subtracting the point representing the SR spike. In Fig. 2 we display our results for the SNR, $R(A,D)=S(\nu_\Omega)/B(\nu_\Omega)$, as a function of D at different values of ν_Ω and A.

The new features of the SNR we detected experimentally can be summarized as follows: (i) the curve $R(A,D)$ vs D peaks at $\Delta V/D=2$, where ΔV is the height of the potential barrier, *independently of the value of A, γ, and ν_Ω*. Moreover, $R(A,D)$ turned out to be proportional to the reciprocal of the conventional bandwidth $\Delta\nu_{\text{expt}}$. (ii) The occurence of SR has been verified for values of ν_Ω as small as 2 Hz (whereas $2\mu_K$ is of the order of 100 Hz or larger) contrary to what was observed in the experiment of Ref. 4. (iii) The maximum value of $R(A,D)$ vs D, $R_{\max}(A)$, increases with A but is quite insensitive to ν_Ω for $\nu_\Omega < \mu_K$ [while $R(A,D)$ flattens out for $\nu_\Omega > \mu_K$]. (iv) The SR mechanism persists for vanishingly small values of the frictional constants (underdamped regime) but $R_{\max}(A)$ strongly depends on γ.[9,10] Most notably, features (ii)-(iv) of SR have been observed also for asymmetric bistable potentials by adding a constant voltage to the external modulating signal $A(t)$.[9]

The theoretical interpretation of SR elaborated in Ref. 2 applies to the discrete process $T(n)$ but fails to reproduce the behavior of $R(A,D)$. For low forcing frequencies, $\nu_\Omega < \mu_K$, the relaxation process (1) can be envisaged as the statistical superposition of two relaxation processes in the asymmetric potentials (adiabatic approximation) $V_\pm(x)=V(x)\pm Ax$ driven by the noise $f(t)$. Since the *small* perturbation $A(t)$ is assumed not to alter the bistable nature of the potential $V(x)$, $V_\pm(x)$ exhibit asymmetric minima in $\pm x_1$ and $\pm x_2$, with $x_1 \approx -x_m - A/2a$. $x_2 \approx x_m - A/2a$, and $x_m=(a/b)^{1/2}$. Note that $V_+(x)=V_-(-x)$ and $V_+(x_1)<V_+(x_2)$. In the adiabatic approximation we can define two Kramers rates, μ_K^+ out of the deeper (metastable) well ($\mu_K^+ < \mu_K$) and μ_K^- out of the shallow one ($\mu_K^- > \mu_K$), respectively. The forcing mechanism alternately tilts $V(x)$ in the configuration V_+ and V_- for half a forcing period so that, if the SR condition $(\mu_K^-)^{-1} < \nu_\Omega^{-1} < (\mu_K^+)^{-1}$ is fulfilled,[2] the output signal switches between the two stable voltages with frequency ν_Ω, giving $x(t)$ no appreciable chance to leave the absolute minimum $\pm x_1$ of $V_\pm(x)$ during one semiperiod π/Ω. The periodicity of the switching signal is blurred by too fast a hopping dynamics, $\nu_\Omega < \mu_K^+$, or a time-dependent perturbation $A(t)$, $\nu_\Omega > \mu_K^-$. This model for SR explains the Ω dependence of $N(T)$. The side peaks detected in Fig. 1(b) can be explained by noticing that in the experimental conditions which maximize the SR effect, μ_K differs from $\mu_K^\pm$ for less than a factor of 2 while $\nu_\Omega \approx \mu_K$. There is a finite probability that $x(t)$ sojourns about the metastable minimum $\pm x_1$ of $V_\pm(x)$ longer than half a forcing period, so that no switch takes place before $V_\pm(x)$ is changed into $V_\pm(-x)$. This implies that $x(t)$ may fluctuate in one semiaxis during an odd π/Ω multiple time interval with exponentially vanishing probability.[9]

The behavior of $R(A,D)$ illustrated in Fig. 2 instead suggests that the SR mechanism persists for any frequency $\nu_\Omega < \mu_K$. The adiabatic approximation can be advocated here to discuss quantitatively the findings of our simulations. In Fig. 3, we plot the normalized autocorrelation function $C(t)$ of the process (1) for some values of A and D. For $\nu_\Omega < \mu_K$ it is possible to distinguish between an exponential relaxation with decay rate[9] $\mu_K^+ + \mu_K^-$ and a periodic branch with frequency[11] ν_Ω. This corresponds to a fast noise-driven relaxation of $x(t)$ towards a periodically modulated steady-state mean value $\langle x(t)\rangle_T$. A good estimate for the amplitude x_T of $\langle x(t)\rangle_T$ in the interesting range of D values, $\Delta V/D > 2$, can be obtained by considering a tilted configuration $V_\pm(x)$ of the potential $V(x)$ and approximating the relevant $x(t)$ distribution function to two normalized δ functions centered at $\pm x_1$ and $\pm x_2$, respectively; i.e.,

$$x_T(A,D) \approx \left| \frac{x_1 e^{Ax_m/D} + x_2 e^{-Ax_m/D}}{e^{Ax_m/D} + e^{-Ax_m/D}} \right| \approx \frac{A}{2a} + x_m tgh \frac{Ax_m}{D} . \qquad (2)$$

In Eq. (2) we made use of the approximate expressions $V_+(x_1) \approx V(x_m) - Ax_m$ and $V_+(x_2) \approx V(x_m) + Ax_m$, only valid for $Ax_m \ll \Delta V = V(0) - V(x_m)$. Approximate expressions for x_1 and x_2 have been given above. As a matter of fact, the amplitude c_T of the periodic branches of $C(t)$ in Fig. 3 agrees with our estimate $c_T = (x_T/x_m)^2$ within a few percent. Under the operating conditions of our analyzer, the relevant SR spike is expressable as

$$S(\nu_\Omega) = c_T x_m^2 / \Delta\nu_{\text{expt}} . \qquad (3)$$

For large-to-intermediate γ values the background of the $x(t)$ power spectrum is closely reproduced by the well-known power spectrum[5,8] of the unperturbed process (1) with $A(t) = 0$,

$$B(\nu) = 2\langle x^2 \rangle \mu_K / (\pi^2 \nu^2 + \mu_K^2) . \qquad (4)$$

Such an assumption is certainly tenable in the limit of low forcing frequency $\nu_\Omega \ll \mu_K$ and small perturbation $Ax_m \ll \Delta V$. In (4) $B(\nu)$ has been normalized to $\langle x^2 \rangle \approx x_m^2$. For $\nu_\Omega \ll \mu_K$ our estimate of R follows immediately[11]:

$$R(A,D) = c_T(A,D)\mu_K / 2\Delta\nu_{\text{expt}} . \qquad (5)$$

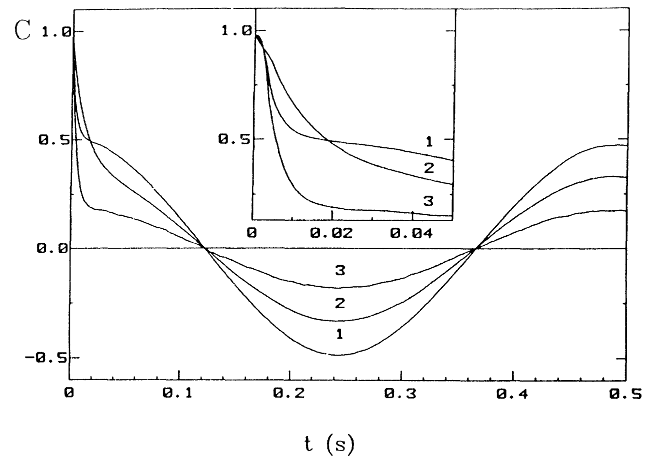

FIG. 3. The output-signal autocorrelation function for $\nu_\Omega = 2$ Hz and (1) $\Delta V/D = 2$, $Ax_m = 0.5\Delta V$; (2) $\Delta V/D = 3$, $Ax_m = 0.22\Delta V$; (3) $\Delta V/D = 2$, $Ax_m = 0.22\Delta V$. The other parameter values are as in Fig. 1. No appreciable statistical error is expected.

In the experimental situation of Fig. 2 with $Ax_m = 0.5\Delta V$ (and $\gamma/\omega_0 = 0.64$) we obtained $c_T = 0.50 \pm 0.01$, $2\mu_K = 280 \pm 5$ Hz, and $R_{\max} = 8.7 \pm 0.4$. This value of $R_{\max}$ is to be compared with our prediction in (5), $R_{\max} = 7.0$. The disagreement between the experimental and theoretical determinations of $R_{\max}$ is expected to decrease with increasing γ. This trend has been verified[9] over a wide range of γ/ω_0 values, $0.06 < \gamma/\omega_0 < 6$. The dependence of $R_{\max}(A)$ on the forcing amplitude is also well reproduced by Eqs. (5) and (2). On recalling that $\mu_K \approx \exp(-\Delta V/D)$, it is also clear why for $Ax_m \ll D$, that $R(A,D)$ peaks at about $\Delta V/D = 2$, as first shown in Ref. 4.

In conclusion, we have characterized quantitatively the notion of stochastic resonance for much wider a class of stochastic systems than previously reported in the literature by including dissipation (and asymmetry[9]) effects. The stochastic resonance mechanism, of potentially frequent occurrence in nature, is explained in detail by means of a simple adiabatic argument.

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[9]L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, to be published.

[10]Peaks corresponding to odd harmonics of the forcing frequency have been clearly resolved (at 3Ω, 5Ω, and 7Ω) as well as in $N(T)$. An observed feature of the experiments in Ref. 4 was the presence of a peak at the second-harmonic frequency. This appears to be in agreement with the actual bidimensional ring-laser model advocated in Ref. 4.

[11]The long-time periodic behavior of $C(t)$ can be determined also by means of a matrix continued fraction expansion of the Fokker-Planck equation associated with the process (1). P. Jung and P. Hanggi, to be published. See also, R. F. Fox, to be published, and B. McNamara and K. Wiesenfeld, to be published.