Anderson Transition in a One-Dimensional System with Three Incommensurate Frequencies

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(Received 31 May 1988; revised manuscript received 9 November 1988)

We report results of numerical investigations of the quantum dynamics of a 1D system subject to a time-dependent perturbation with three incommensurate frequencies. These results demonstrate a transition from localized to extended states occurring at a critical value of the perturbation parameter. The dependence of the localization length and of the diffusion rate on this parameter near the critical point is analyzed and found to be in agreement with the predictions of renormalization-group theory.

PACS numbers: 05.45.+b, 72.15.Rn, 73.20.Jc

The investigation of quantum systems, which in the classical limit become chaotic, has shown that, in the case of 1D systems subject to perturbations periodic in time, quantum effects lead to localization of diffusive excitation.[1] This phenomenon is similar to the Anderson localization for a particle in a 1D lattice with a disordered potential[2]; it should be stressed, however, that in our case no external random element is introduced since the perturbation is periodic and that localization is related here to quasienergy eigenfunctions and occurs in momentum instead of configuration space.

Further investigations showed that the introduction of a second, incommensurate perturbing frequency produces a sharp increase in the localization length, which grows exponentially with the quasiclassical diffusion rate.[3] Again, this is close to the prediction of the theory of weak localization in 2D disordered lattices.[4-6] In the same vein, a 2D time-dependent model has also been studied in Ref. 7.

In this paper we investigate a time-dependent model with three incommensurate frequencies. Since new, incommensurate frequencies in the time-dependent problem introduce new dimensions in the extended phase space, such a model is expected to correspond to a 3D lattice problem, where a transition from localized to extended states occurs at some critical parameter value.

We were indeed able to observe such a transition from localized to delocalized behavior in our model. By analyzing this transition, we obtained some indications that may be helpful in clarifying the nature of the Anderson transition itself, which is still in discussion in solid-state physics.[8,9] In this connection, we wish to emphasize that the 1D character of our time-dependent model allows for a sharp reduction of the computation time needed to analyze the transition, so that recourse to scaling assumptions can be avoided.

Our model is a variant of the kicked rotator model.[1-3] We consider the motion of a particle on a circle, described by the time-dependent Hamiltonian:

$$H=H_0+V(\theta,t)\sum_{s=-\infty}^{+\infty}\delta(t-s)\,. \tag{1}$$

The second term describes kicks occurring periodically in time with period one. The free evolution between kicks is given by the Hamiltonian H_0:

$$H_0|n\rangle=E_n|n\rangle,\quad |n\rangle=e^{in\theta}/(2\pi)^{1/2}\,. \tag{2}$$

We assume the eigenvalues E_n to be random numbers uniformly distributed in $(0,2\pi)$. Unlike for the usual kicked rotator, we also assume V to explicitly depend on time according to

$$V\equiv V(\theta,\theta_1+\omega_1 t,\theta_2+\omega_2 t)\,, \tag{3}$$

with V a periodic function of its three arguments to be specified later, and θ_1 and θ_2 are arbitrarily prescribed phases. We would like ω_1 and ω_2 to be incommensurate with each other and also with the frequency of the kicks. Therefore, we take $\omega_1=2\pi\lambda^{-1}$ and $\omega_2=2\pi\lambda^{-2}$, with $\lambda\sim 1.3247\ldots$ the real root of the cubic equation $x^3-x-1=0$. With this choice, ω_1 and ω_2 are a "most incommensurate" pair of numbers.[10] Thus (1) describes the motion of a rotator subjected to periodic kicks, the strength of which is modulated in time by the frequencies ω_1 and ω_2.

The evolution of this rotator, from just after one kick to just after the next, is given by (we take $\hbar=1$)

$$\psi(\theta,t+1)=e^{-iV(\theta,t+1)}e^{-iH_0}\psi(\theta,t)\,. \tag{4}$$

This formation of the rotator dynamics is very convenient for numerical simulations because the time dependence of V is explicitly known. Nevertheless, in order to elucidate the connection of this time-dependent problem with a 3D tight-binding model, we must resort to a different formulation as follows. First of all, we consider the phases θ_1 and θ_2 as new dynamical variables,

with conjugates momenta n_1 and n_2. Then we consider the Hamiltonian

$$H' = H_0(\hat{n}) + \omega_1\hat{n}_1 + \omega_2\hat{n}_2 + V(\theta,\theta_1,\theta_2)\sum_{s=-\infty}^{+\infty}\delta(t-s)\,, \quad (5)$$

with $\hat{n}_{1,2} = -i\,\partial/\partial\theta_{1,2}$. Equation (5) describes a quantum rotator with three freedoms $(\theta,\theta_1,\theta_2)$ subjected to periodic kicks, the strength of which is not explicitly time dependent. The one-period propagator for this rotator is the unitary operator

$$e^{-iV(\theta,\theta_1,\theta_2)}\,e^{-i[H_0(\hat{n})+\omega_1\hat{n}_1+\omega_2\hat{n}_2]}\,.$$

In order to show that the 3D quantum model defined by (5) and the 1D model defined by (1) and (3) are substantially equivalent, we rewrite the Schrödinger equation for the 3D model

$$i\,d/dt\,\psi(\theta,\theta_1,\theta_2,t) = H'\psi(\theta,\theta_1,\theta_2,t)\,,$$

in the interaction representation defined by

$$\psi(\theta,\theta_1,\theta_2,t) = e^{-i(\omega_1\hat{n}_1+\omega_2\hat{n}_2)t}\tilde{\psi}(\theta,\theta_1,\theta_2,t)\,. \quad (5a)$$

In this way we obtain

$$i\,d\tilde{\psi}/dt = H_0\tilde{\psi} + V(\theta,\theta_1+\omega_1 t,\theta_2+\omega_2 t)\sum_{s=-\infty}^{+\infty}\theta(t-s)\tilde{\psi}\,,$$

i.e., the Schrödinger equation for the evolution of the 1D model.

We can now apply a transformation to the three rotator (5), which was originally devised in Ref. 2 for the standard, 1D kicked rotator and was subsequently generalized in Ref. 11. Because of this transformation the problem of our determining the quasienergy eigenvalues and eigenvectors for the three rotator turns out to be formally equivalent to solving the equation

$$T_{\mathbf{n}}u_{\mathbf{n}} + \sum_{r\neq 0} W_{\mathbf{r}}u_{\mathbf{n}+\mathbf{r}} = \epsilon u_{\mathbf{n}}\,, \quad (6)$$

where $\mathbf{n}\equiv(n,n_1,n_2)$ and $\mathbf{r}$ label sites in a 3D lattice,

$$T_{\mathbf{n}} = -\tan[\tfrac{1}{2}(E_n + n_1\omega_1 + n_2\omega_2 + \lambda)]\,,$$

λ is quasienergy, $W_{\mathbf{r}}$ are coefficients of a threefold Fourier expansion of $\tan[\frac{1}{2}V(\theta,\theta_1,\theta_2)]$, and $\epsilon = -W_0$.

We now choose

$$V(\theta,\theta_1,\theta_2) = -2\tan^{-1}[2k(\cos\theta+\cos\theta_1+\cos\theta_2)]\,, \quad (6a)$$

so that (6) becomes

$$T_{\mathbf{n}}u_{\mathbf{n}} + k\sum_{\mathbf{r}}{}'u_{\mathbf{r}} = 0\,, \quad (7)$$

where the sum $\sum'$ includes only the nearest neighbors to $\mathbf{n}$. The tight-binding model (7) with the potential $T_{\mathbf{n}}$ is in a sense equivalent to the original rotator problem. The quasienergy eigenfunctions of the rotator will be localized or extended over the unperturbed eigenstates of H_0, depending on whether the tight-binding model has localized or extended eigenstates; in the localized case, the localization length will be the same. Since the dynamics of the rotator is determined by the nature of its quasienergy eigenstates, any change from localized to extented states that may take place in the tight-binding model (7), as the coupling parameter k is increased, will be mirrored by a simultaneous change in the rotator dynamics, from a localized recurrent behavior to an unending spreading over the unperturbed base. As we mentioned above, the latter type of transition can be numerically detected with less effort than by our directly tackling the tight-binding model.

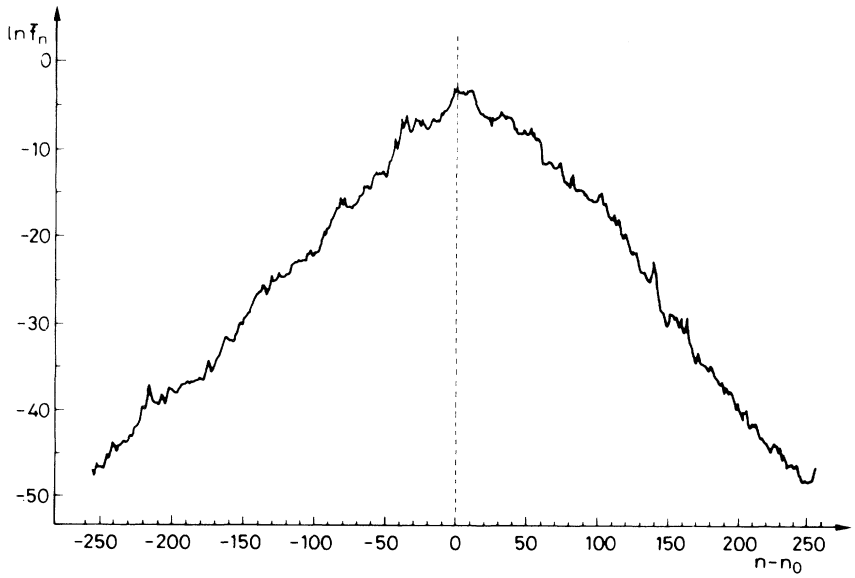

FIG. 1. Example of localized steady-state probability distribution $\bar{f}_n$ over the unperturbed levels averaged over 5000 iterations within the interval $95\,000 < t < 100\,000$. Here parameter $k = 0.38$.

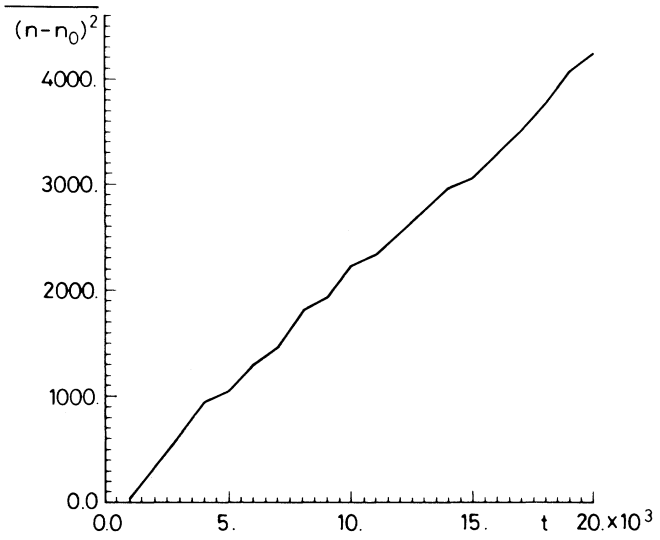

FIG. 2. Second moment $\langle (n-n_0)^2 \rangle$ of the probability distribution as a function of time t (in number of iterations) in the delocalized regime. Here parameter $k=0.6$.

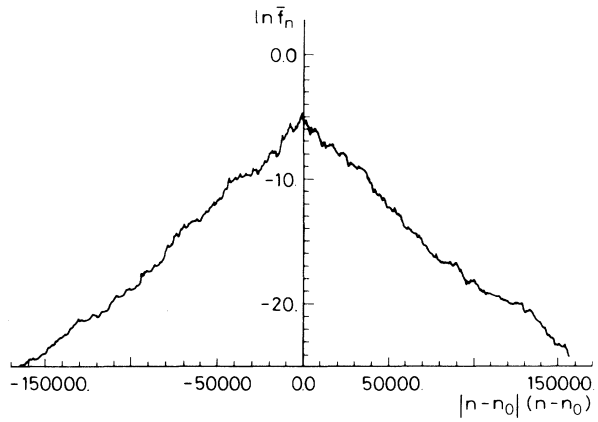

FIG. 3. Example of a Gaussian probability distribution $\bar{f}_n$ over the unperturbed levels for the same case of Fig. 2. The probability distribution is here averaged over 1000 iterations within the interval $19000 < t < 20000$.

The disorder in the model (7) is given by the pseudorandom character of the potential $T_{\mathbf{m}}$. We remark that by replacing $n_1\omega_1$ and $n_2\omega_2$ by random numbers, one would get a 3D Lloyd model. In that case, however, the time dependence of V could not be made explicit as in Eq. (4). One should then numerically simulate a 3D rotator rather than a 1D one, with a consequent sharp reduction of the allowable number of iterations.

The model was investigated by numerical simulation of the quantum dynamics defined by (1) with phases $\theta_1=\theta_2=0$. A basis of unperturbed eigenstates up to 512 was used. The initial state was chosen in the middle of this base and its time evolution was numerically determined by iteration of the quantum map (4) giving the one-period evolution, up to 10^5 iterations, for different values of the perturbation parameter k. A transition between two different types of motion was observed around a value $k_{\mathrm{cr}} \sim 0.47$, with localization occurring for $k < k_{\mathrm{cr}}$ and unbounded diffusion taking place for $k > k_{\mathrm{cr}}$. In the localized regime, the time-averaged steady-state distributions were found to decay exponentially with the level number (Fig. 1), so that the corresponding average localization lengths could be determined by our fitting the probability distributions with the exponential law $\exp(-2|n-n_0|/l)$. In the delocalized regime, unbounded diffusive excitation occurs and the related diffusion rates were determined by two ways; first, from the time dependence of the second moment of the probability distributions (Fig. 2), and, second, by our fitting the probability distributions with the Gaussian form which would be predicted by the diffusion law (Fig. 3).

By this second fitting we also obtained the number of levels which were effectively involved in the diffusion. In all cases, these levels were found to be a fraction close to 1 of the total number of levels, i.e., practically all states were delocalized. This was further confirmed by the closeness of the values of the diffusion coefficients obtained according to the two above described methods.

In Fig. 4 we show the dependence of the diffusion rate $D=\langle (n-n_0)^2 \rangle/t$ (in the delocalized regime) and of the inverse localization length $\gamma=l^{-1}$ (in the localized regime) on the perturbation parameter k. In order to suppress fluctuations, for each fixed value of k the values of γ and D were computed for ten different realizations of the random spectrum of H_0, and average values were

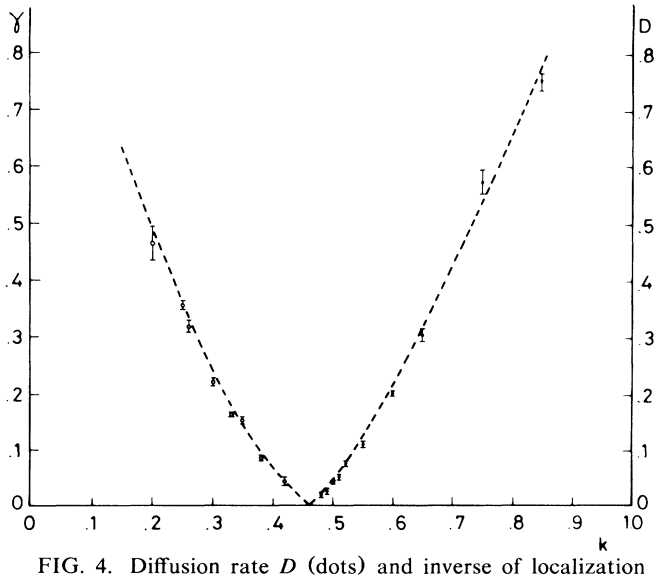

FIG. 4. Diffusion rate D (dots) and inverse of localization length $\gamma=1/l$ (circles) as a function of perturbation parameter k. Error bars were obtained from statistics over ten different realizations of the random spectrum. The dotted lines result from a three parameters least-squares fit (MINUIT) of numerical data.

then found. The error bars give the standard deviations. It is apparent that the transition took place near $k_{cr} \sim 0.465$. The dependences of D and γ near k_{cr} are consistent with power laws, $D \sim D_0(k-k_{cr})^s \gamma \sim \gamma(k-k_{cr})^\nu$. By a three parameters least-squares fit of the dependence of D, we obtained $D_0 \sim 2.5$, $k_{cr} \sim 0.46$, and $s \sim 1.25$ with a $\chi^2 = 10$. Errors ΔD_0, Δk_{cr}, and Δs (corresponding to a change of χ^2 equal to 1) were, respectively, 1, 0.0044, and 0.04. An analogous fit for the dependence of γ gave $\gamma_0 \sim 3.5$, $k_{cr} \sim 0.469$, and $\nu \sim 1.5$ with $\chi^2 = 24$ and errors $\Delta k_{cr} = 10^{-3}$, $\Delta \gamma_0 = 3 \times 10^{-2}$, and $\Delta \nu \sim 10^{-2}$.

Thus the two fittings give very close values of k_{cr}. Renormalization theory predicts an exponent equal to one for the behavior of the inverse of localization length close to the transition point. The same exponent should describe, in the delocalized regime, the dependence of conductivity on $k - k_{cr}$.[6,8] The same dependence should therefore be expected for the diffusion coefficient.

The numerically obtained values are consistent with these predictions. They are both close to 1, in agreement with Ref. 12, according to which $s = \nu = 1 + O(\epsilon^4)$; $\epsilon = d - 2$, with d the lattice dimension. The only approximate agreement of numerical values with theoretical predictions is related to the fact that for values of k very close to k_{cr}, a number of iterations much larger than $l^3 \sim (k - k_{cr})^{-3}$ is needed in order to distinguish localized from diffusive behavior and to get very accurate values for γ and D.

An interesting remark is that the value of k_{cr} is not the same as for the Lloyd's model. In that case, a numerical determination of k_{cr} by the method of Lyapunov's exponents yields a value ≈ 0.2.[13]

On account of the above described results, we have here an example of a 1D system subject to a time-dependent perturbation with three incommensurate frequencies,[3] in which a transition from localized to diffusive behavior occurs. Together with previous results for a model with two incommensurate frequencies, the results above indicate a definite correspondence between the number of incommensurate frequencies and the lattice dimension in solid-state problems. This introduces a possibility of relatively easy numerical investigations of localization and Anderson transition in higher dimension also.

We are greatly indebted to B. V. Chirikov for his attention at this work. This work was performed with the support of Consiglio Nazionale delle Ricerche (Italy).

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