Fractal Structure of Ising and Potts Clusters: Exact Results

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It is shown that previously defined clusters, which give a geometrical description of the fluctuations in the q-state Potts model, at criticality have a fractal structure made of links and blobs as in percolation. Using the mapping from the Potts model to the Coulomb gas it is found that the fractal dimension of the links or red bonds D_R is given by $\frac{5}{4}$, $\frac{3}{4}$, $\frac{13}{24}$, $\frac{7}{20}$, 0, while the fractal dimension of the external hull D_H is given by 2, $\frac{7}{4}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{3}{2}$, for $q=0,1,2,3,4$. A model originally introduced by Mandelbrot and Given for percolation clusters if found to correctly describe the fractal structure of the Potts clusters.

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How to characterize geometrically a fluctuation near a critical point is a long-standing problem[1-8] that recently has received renewed attention, due to a novel experiment in which direct visual observation of critical fluctuations was possible.[9] It is now well established that in an Ising model the naive definition of clusters made of nearest-neighbor parallel spins is not satisfactory.[2] These clusters are in fact too large, representing both correlations and pure geometrical effects. As an example at $T=\infty$ in a typical configuration clusters of parallel spins are present although there is total absence of correlations. To eliminate the pure geometrical effect a different definition of cluster was proposed[4] for the Ising model and generalized[5] to the q-state Potts model. These new clusters are defined as nearest-neighbor sites in the same state connected by bonds, each bond being present with probability $p=1-e^{-K}$, where K is the nearest-neighbors (NN) coupling constant of the Potts Hamiltonian $-\mathcal{H}_P/kT=K\sum_{\langle ij\rangle}\delta_{\sigma_i\sigma_j}$ with $\sigma_i=1,2,\ldots,q$ being the spin variables (Fig. 1). With such definition,[4,5] the clusters behave correctly at the critical point T_c; namely, their radius and the density of sites in the infinite cluster behave critically as functions of $|T-T_c|$, respectively, with the correlation length exponent $\nu(q)$ and the order-parameter exponent $\beta(q)$ of the q-state Potts model. In particular, these clusters coincide with the clusters of the tree percolation problem for $q=0$, of the random percolation problem for $q=1$, and of the Ising model for $q=2$.[3] It has been shown[4,5] that the statistics of these clusters can be obtained from the following Hamiltonian:

$$-\mathcal{H}/kT=J\sum_{\langle ij\rangle}(\delta_{\tau_i\tau_j}-1)\delta_{\sigma_i\sigma_j}+K\sum_{\langle ij\rangle}\delta_{\sigma_i\sigma_j}, \qquad (1)$$

where the second term, which controls the distribution of the spin variables, is the q-state Potts Hamiltonian, whereas the first term contains auxiliary Potts variables $\tau_i=1,2,\ldots,s$ and controls the bonds distribution. More specifically it can be shown[4,5] that by differentiating the free energy of Hamiltonian (1) and taking the limit $s=1$ one obtains the distribution of the clusters made of sites in the same spin configuration connected by bonds, present with probability $p=1-e^{-J}$. By choosing the particular bond probability with $J=K$, Hamiltonian (1) assumes a simple form and describes the correct clusters. These are the only clusters that I will consider throughout the paper.

The interest in the geometrical description of the fluctuation has gone much beyond the original motivation. Recently, Swendsen and Wang[10] (SW) introduced a new dynamics, which instead of flipping one spin at a time, flips altogether all the spins in the same cluster. The resulting algorithm has proved to be extremely fast compared with the Glauber dynamics, with a drastic reduction of the critical slowing down. Using geometrical concepts, a scaling *Ansatz* was recently proposed,[11] relating the exponents of the two dynamics. How to elaborate the SW dynamics to further reduce the critical slowing down is a big open question which may be enlightened by

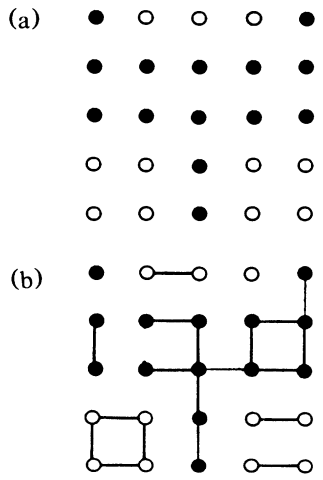

FIG. 1. (a) Ising configuration: "up" and "down" spins are represented, respectively, by filled and empty circles. (b) Correct clusters obtained from the configuration given in (a) by putting bonds between sites in the same state, with probability $p=1-e^{-K}$. Note the spanning cluster of the "up" spins is made of three dangling bonds, four red bonds (if one of these is cut the cluster does not span anymore), and one blob made of four bonds.

precise information on the fractal structure of the Potts clusters.

In percolation much progress was done when it was recognized that the backbone of the incipient infinite cluster at p_c has a fractal structure made of links and blobs. To be more precise, in a typical configuration which spans the system from top to bottom one has to distinguish dangling bonds and a backbone. The backbone is made of singly connected bonds, called red bonds, such that if one is cut, the top is disconnected from the bottom and the blobs which are made of multiply connected bonds.[12,13] While the fractal dimension of the whole cluster[12] is identical to the magnetic scaling exponent $y_H(1)$, the fractal dimension of the red bonds[13] is identical to the thermal scaling exponent $y_K(1)$. How much of this picture can be extended to the clusters of the q-state Potts model has so far been an open problem.

It has already been noted[4,5] that at T_c these clusters are fractals with fractal dimension[14] $D(q)=d-\beta(q)/\nu(q)$, where d is the Euclidean dimension. Note that $D(q)$ coincides with the magnetic scaling exponent $y_H(q)$ and reduces to the well-known result for percolation for $q=1$.

It is interesting to note that the exact value[15] of $D(q)$ does not vary substantially with q, as shown in Table I for $d=2$. This observation can be understood by noting that using this geometrical approach, the driving mechanism of the critical behavior can be viewed as coalescence of clusters just as in a percolation transition. Then one would expect for any q that the fractal dimension should be close to the fractal dimension of the critical clusters in the percolation problem. This also explains the observation, known as strong universality,[16] made by Suzuki that over a large class of models the ratio γ/ν or β/ν does not vary appreciably. Since these ratios of critical exponents for fixed d depend only on the magnetic scaling exponent, which is identical to the fractal dimension, the strong universality is a consequence of the quasiuniversal feature of the fractal dimension as discussed above.

It is the aim of this paper to investigate how much the fractal structure of the percolation picture can be extended to the q-state Potts clusters and, by exact results in 2D, to see what are the geometrical properties which change substantially with q, in particular, when q approaches the critical value q_c above which the transition is of first order.

Links and blobs.—At T_c the clusters of the q-state Potts model are found to have a structure made of links and blobs just as in percolation [Fig. 1(b)]. However, the fractal dimension of the red bonds does not coincide with the thermal scaling exponent $y_K(q)$. Instead, it is found to coincide with the bond probability scaling exponent $y_J(q)$ associated to the variable J in the Hamiltonian (1)

$$D_R(q)=y_J(q)\,. \tag{2}$$

To be more precise, $y_J(q)=\nu_B^{-1}(q)$ where $\nu_B(q)$ is the critical exponent associated with the divergence of the radius of the clusters in which the temperature is fixed at its critical value K_c and the bond probability p as an independent variable approaches from below $p_c=1-e^{-K_c}$; i.e., $\xi\sim(p_c-p)^{-\nu_B(q)}$. The result of Eq. (2) is based on the following relation:

$$p\,dp_{ij}/dp=\langle\lambda_{ij}\rangle\,, \tag{3}$$

where $p_{ij}=\langle\gamma_{ij}\rangle$ is the probability that i and j are connected and γ_{ij} is 1 if i and j are the same Potts cluster, 0 otherwise, and λ_{ij} is the number of red bonds between i and j. The brackets denote the average over all Potts variables and bond configurations; namely,

$$\langle\cdots\rangle=\frac{1}{Z}\sum_{\sigma_i}e^{-\beta\mathcal{H}}\sum_{G}\cdots p^{B}(1-p)^{D}\,, \tag{4}$$

where $Z=\sum_{\sigma_i}e^{-\beta\mathcal{H}}$, the first sum is over all Potts variables, the second sum is over all graph G contained in the graph $E\{\sigma_i\}$ made of all bonds connecting NN sites in the same state, for a fixed configuration $\{\sigma_i\}$ B is the number of bonds in the graph G, and D is the number of empty bonds. In Eq. (3) the derivative is made with respect to the bond probability, considered as an independent variable, and then p is put equal to $1-e^{-K}$. Relation (3) follows from a general result[13] valid for random-bond percolation for any graph, which relates the pair connectedness to the average number of red bonds as in (3). Therefore by writing such a relation for any graph $E\{\sigma_i\}$ and averaging over all configurations $\{\sigma_i\}$ we obtain a relation (3) valid for Potts clusters. From (3) following the same procedure as for random percolation[13] it follows Eq. (2).

Hull and bridge bonds.—The typical nonspanning configurations are characterized by almost spanning clusters. For each cluster the hull or external perimeter is made of all the absent bonds surrounding the cluster. The absent bonds are of two types, those between sites in the same state and those between sites in different states. The "bridges"[17] instead are absent bonds such that if one was present two clusters would merge in one single spanning cluster. The absent bonds that we consider here are only those between two sites in the same state. It is possible to show that the fractal dimension of the red bonds and of the bridge bonds is identical, $D_B(q)=D_R(q)$. This is a generalization of a result valid in percolation[17] ($q=1$) and follows from a relation similar to (3)

$$(1-p)\,dp_{ij}/dp=\langle\mu_{ij}\rangle\,, \tag{5}$$

where μ_{ij} is the number of bridge bonds between the clusters to which i and j belong, such that if one was present the two sites would be connected. Since relation (5) has been proved for random percolation defined on any graph, using the same reasoning as for the red bonds, relation (5) can be extended to the Potts clusters

and therefore as for random percolation[17] it follows $D_B(q)=D_R(q)$. For $d=2$ and $q=1$ (percolation) we know the exact value of $D_R(1)=\frac{3}{4}$ and the exact[18] value of the fractal dimension of the hull $D_H(1)=\frac{7}{4}$ previously conjectured in Ref. 19. The next question is to find the values of $D_R(q)$ and $D_H(q)$ for all values of q.

Exact values for $D_R(q)$and $D_H(q)$.—It is possible to identify $D_R(q)$ and $D_H(q)$ with some of the exponents obtained by Saleur and Duplantier[18] by mapping the Potts model on the Coulomb gas. The result is the following:

$$D_H(q)=(g+2)/g\,, \tag{6}$$

$$D_R(q)=(8g-3g^2+16)/8g\,, \tag{7}$$

with $q=2+2\cos(\pi g/2)$, $g\in[2,4]$. $D_H(q)$ and $D_R(q)$ as functions of q are plotted in Fig. 2 and their values for discrete $q=0,1,2,3,4$ are given in Table I. For $q=1$ Eqs. (6) and (7) give the well-known results for random-bond percolation.[18] So far no results are known for the other exponents except for the fractal dimension of the red bonds of the correct Ising clusters. This has been calculated by de Arcangelis,[20] who found, using Monte Carlo methods, $D_R(2)=0.52+0.03$ in good agreement with the exact result 0.5417. Very recently it has been conjectured[21] on the base of conformal invariance $y_J(2)=\frac{13}{24}$ which in view of (2) coincides with $D_R(2)$. Another conjecture[5] based on the Migdal renormalization group gave $y_J(4)=D_R(4)=0$. Both conjectures are confirmed by the exact results of Table I.

From Fig. 2 and Table I it appears that, unlike the fractal dimension of the whole cluster, $D_R(q)$ and $D_H(q)$ do change substantially and characterize the different models as functions of q. Particularly sensitive to q is the fractal dimension of the red bonds, which has its largest value at $q=0$ (tree percolation), where the backbone is made only of links. As q approaches q_c the cluster becomes less ramified until the red bonds vanish $[D_R(4)=0]$. This results in a drastic structural change from a links and blobs picture to only blobs, anticipating a first-order transition.

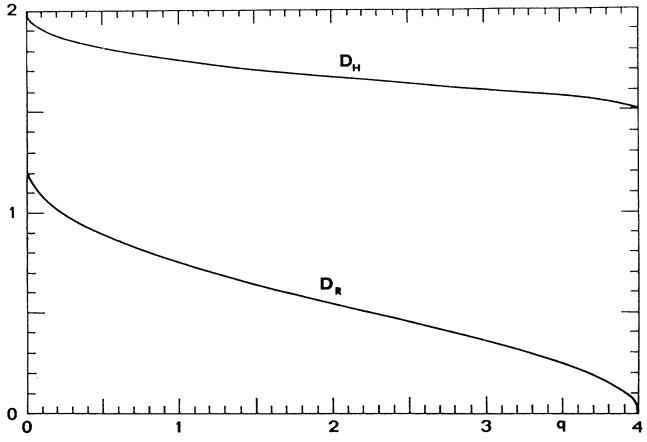

FIG. 2. Fractal dimensions of the singly connected bonds (red bonds) $D_R(q)$ and the hull $D_H(q)$ for the correct clusters (see Fig. 1) as function of q.

To find the exact results (6) and (7) let us consider the correlation function $g_1(x-y)=\langle\theta_1(x,y)\rangle$, where x and y are the middle-point coordinates of a bond and $\theta_1(x,y)$ is 1 if x and y are on the hull of the same cluster, 0 otherwise. The average is done according to (4). By taking into account that for any configuration $\{\sigma_i\}$, $e^{-\beta\mathcal{H}}=3^{-KD'}$, where D' is the number of bonds between NN sites in different states, and by averaging over the spin configurations it easily follows that $g_1(x-y)=A_1(x-y)$, where

$$A_1(x-y)=Z^{-1}\sum_{G_1}\theta_1(x,y)w(G_1)\,,$$

where the sum is over all graph G_1 contained in the *whole* lattice and $w(G_1)=p^B(1-p)^Dq^C$, B is the number of bonds in G_1, D is the number of absent bonds ($B+D=$total number of bonds in the lattice), and C is the number of connected components. Saleur and Duplantier[18] have shown that at criticality $A_1(x-y)\sim|x-y|^{-2x_1}$ with $x=(g-2)/g$; therefore, $g_1(x-y)$ decays also in the same way, and applying standard procedures[18] one finds that $D_H(q)=2-x_1$ from which result (7) follows.

Similarly we define $g_2(x-y)=\langle\theta_2(x,y)\rangle$, where $\theta_2(x,y)=\theta_2^{AA}(x,y)+\theta_2^{AB}(x,y)$ with $\theta_2^{AA}(x,y)=1$ if x and y are empty bonds bridging two clusters made of sites in the same state (these are bridge bonds), 0 otherwise, and $\theta_2^{AB}(x,y)$ is 1 if x and y are empty bonds bridging two clusters made of sites in different states. With obvious notation we can write $g_2(x-y)=g_2^{AA}(x-y)+g_2^{AB}(x-y)$. Taking the sum over all spin configurations it is possible to show that $g_2^{AA}(x-y)=[(q-1)/q]g_2(x-y)$; $g_2^{AB}(x-y)=q^{-1}g_2(x-y)$, and $g_2(x-y)=A_2(x-y)$ with $A_2(x-y)=\sum_G\theta_2(x,y)\times W(G)$. Saleur and Duplantier[18] have shown that at criticality $A_2(x-y)\sim|x-y|^{-2x_2}$, where $x_2=1+\frac{3}{8}g+2/g$. Therefore the fractal dimension of the bridge bonds which is related to $g^{AA}(x,y)$ for $q>1$ and to $g(x,y)$ for $q=0,1$ has a fractal dimension $D_R(q)=2-x_2(q)$, from which follows Eq. (6).

Fractal model.—Mandelbrot and Given[22] (MG) pro-

TABLE I. Fractal dimensions of the whole cluster (D), of the hull (D_H), and of the red bonds (D_R).

q	D	D_H	D_R
0	2	2	$\frac{5}{4}$
1	$\frac{91}{48}$	$\frac{7}{4}$	$\frac{3}{4}$
2	$\frac{15}{8}$	$\frac{5}{3}$	$\frac{13}{24}$
3	$\frac{28}{15}$	$\frac{8}{5}$	$\frac{7}{20}$
4	$\frac{15}{8}$	$\frac{3}{2}$	0

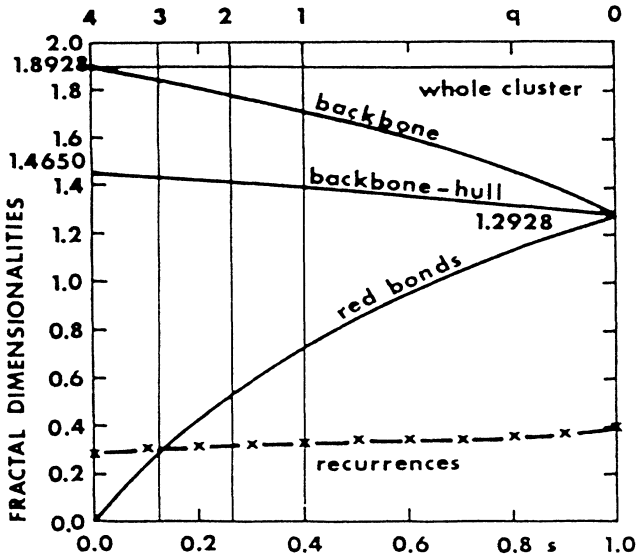

FIG. 3. Adapted from Fig. 3 of Ref. 22. Fractal dimensionalities for different quantities as a function of the parameter s in the Mandelbrot-Given model. The vertical lines correspond to the values of s which map onto $q=0,1,2,3,4$, as explained in text.

posed a fractal model for percolation clusters in 2D. However, this model, as we will see, turns out to be much richer than expected and can describe Potts clusters besides percolation clusters. The MG model is constructed from a Sierpinsky gasket of base $b=3$ in which the order of ramification is controlled by a parameter s, in a range of values between 0 and 1. Mandelbrot and Given calculated the fractal dimension of various quantities as functions of s, such as red bonds, backbone, and backbone hull. To describe the percolation cluster they chose $s=0.4$ which gave results that better fitted the available data.

Now using the exact results obtained in this paper for $D(q)$, it is possible to find a correspondence between the whole range of values of s and the parameter q of the q-state Potts model. This can be done using the implicit equation $D_R^{\mathrm{MG}}(s)=D_R(q)$, where $D_R^{\mathrm{MG}}(s)$ is the fractal dimension of the red bonds as a function of s in the MG model (Fig. 3).

One can check the validity of this mapping by comparing the available information in the two models: (1) The fractal dimension of the whole cluster in the MG model is 1.8298 for all values of s and has a value of about 1.9 in the Potts model for all values of q; (2) the fractal dimension of the hull for $s=4$ is 1.465 which compares well with $D_H(0)=1.5$; and (3) for $s=1$ and $q=0$ where both models are loopless we have $D_R^{\mathrm{MG}}(1)=1.293$ which compares well with $D_R(0)=1.25$.

In conclusion, I have shown that the fractal structure of the Potts clusters is made of links and blobs as in percolation. The fractal dimension of the clusters coincides with the magnetic scaling field $y_H(q)$, whereas the fractal dimension of the red bonds coincides with the bond probability scaling exponent $y_J(q)$. This exponent is distinct from the thermal scaling exponent $y_K(q)$ except for $q=1$. What is the geometrical quantity related to the thermal scaling exponent for $q\neq 1$ is still an open problem. We finally remark that the fractal dimension of the red bonds goes to zero at $q_c=4$, showing a drastic change in the fractal structure due to the appearance of the first-order transition, whereas $y_H(q)$ is roughly constant and $y_K(q)$ is an increasing function of q, without showing any peculiarity at q_c.

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