Higher-Order Corrections in (2+1)-Dimensional QED

Daniel Nash
Physics Department, Yale University, New Haven, Connecticut 06511
(Received 7 April 1989)

QED in 2+1 dimensions, analyzed to lowest order in the $1/N$ expansion, has been shown to exhibit dynamical chiral-symmetry breaking when the number N of fermions is less than $32/\pi^2$. This analysis is extended by considering higher-order corrections to the gap equation. It is shown that while these corrections modify the form of the gap equation, the nature of the symmetry breaking remains the same. It is also shown that the critical number of fermions is a gauge-invariant quantity.

PACS numbers: 11.15.Pg, 11.30.Qc, 12.20.Ds

The model we wish to consider is quantum electrodynamics in 2+1 dimensions. It has the following Lagrangian:

$$L=\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/)\psi-\tfrac{1}{4}F_{\mu\nu}^2\,, \tag{1}$$

where we have N four-component spinors ψ. This theory is super-renormalizable in 2+1 dimensions. However, the same power-counting arguments show that, in the massless case, there are ir divergences that become worse as the uv behavior gets better. To remedy this we will use the $1/N$ expansion.[1-3] This will soften the ir behavior of the photon propagator and leave the Green's functions finite order by order. Since the theory is massless, the mass scale is set by the dimensional coupling constant $\alpha=Ne^2/8$ which is kept fixed as $N\to\infty$. Everything in the theory is rapidly damped for momentum scales $p>\alpha$.

To study chiral-symmetry breaking we use four-component spinors. This allows us to introduce γ_3 and γ_5 which anticommute with γ_0, γ_1, and γ_2 in the Lagrangian. The massless theory is then invariant under $\psi\to\exp(i\alpha\gamma_3)\psi$ and $\psi\to\exp(i\beta\gamma_5)\psi$. Together with the identity matrix and $[\gamma_3,\gamma_5]$ we have a global U(2N) "chiral" symmetry. A mass term $m\bar{\psi}\psi$ will break this symmetry to U(N)×U(N). One may also consider a parity-violating mass term $m\bar{\psi}\frac{1}{2}[\gamma_3,\gamma_5]\psi$ but we will not do so here.[4]

Following Ref. 5 we study solutions of the Dyson-Schwinger gap equation. The inverse fermion propagator is $S(p)^{-1}=-p\!\!\!/[1+A(p)]+\Sigma(p)$, where $A(p)$ is the wave-function renormalization and $\Sigma(p)$ is a dynamically generated flavor-independent fermion mass. The Dyson-Schwinger gap equation is

$$\Sigma(p)=\frac{2\alpha}{N}\mathrm{Tr}\int\frac{d^3k}{(2\pi)^3}\,\frac{\gamma^\mu D_{\mu\nu}(p-k)\{k\!\!\!/[1+A(k)]+\Sigma(k)\}\Gamma^\nu(p,k)}{k^2[1+A(k)]^2+\Sigma^2(k)}\,, \tag{2}$$

where, in the Landau gauge,

$$D_{\mu\nu}(p-k)=\frac{g_{\mu\nu}-(p-k)_\mu(p-k)_\nu/(p-k)^2}{(p-k)^2[1+\Pi(p-k)]} \tag{3}$$

and $\Gamma^\nu(p,k)$ is the vertex. The lowest-order approximations in $1/N$ are $A(p)=1$, $\Pi(p-k)=\alpha/|p-k|$, and $\Gamma^\nu(p,k)=\gamma^\nu$, where we have assumed the fermion mass is negligible in the calculation of $\Pi(p-k)$. After angular integration this yields

$$\Sigma(p)=\frac{4\alpha}{\pi^2Np}\int_0^\infty dk\,\frac{k\Sigma(k)}{k^2+\Sigma^2(k)}\ln\left(\frac{k+p+\alpha}{|k-p|+\alpha}\right). \tag{4}$$

A numerical and analytical study of this equation was carried out in Ref. 5 yielding a critical number of fermions N_c, such that for $N>N_c$, $\Sigma(p)=0$. It is argued there that since the theory is strongly damped for $p>\alpha$, it is reasonable to assume that the relevant physics occurs when $p/\alpha<1$. Hence, only the lowest-order terms in p/α are kept, and a hard cutoff at $p=\alpha$ is imposed. Also, by choosing N near N_c, $\Sigma(k)$ can be made arbitrarily small and the region $k\leq\Sigma(k)$ in the integral can be neglected. This agrees with the above approximation for $\Pi(p-k)$. Then, linearizing the kernel, we find

$$\Sigma(p)=\frac{8}{\pi^2N}\int_0^\infty dk\,\frac{\Sigma(k)}{\max(k,p)}\,. \tag{5}$$

If we set $\Sigma(p)=p^b$ we have

$$b(b+1)=-8/\pi^2N \tag{6a}$$

or

$$b=-\frac{1}{2}\pm\frac{1}{2}\left(1-\frac{32}{\pi^2N}\right)^{1/2}. \tag{6b}$$

As discussed in Ref. 5, chiral-symmetry breaking occurs when b becomes complex, that is for $N<N_c$, where $N_c=32/\pi^2\approx 3.2$.

The existence of the critical value N_c has been obtained using only the leading terms in the $1/N$ expansion of the kernel. It is important to examine whether this result is reliable given the smallness of N_c. Some evidence, both for the qualitative conclusions that an N_c exists and

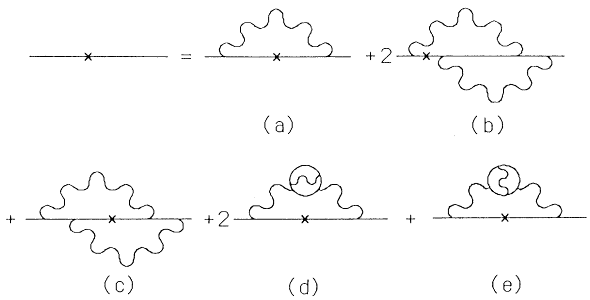

FIG. 1. The diagrams that contribute to order $1/N^2$ in the expansion of the linearized kernel of the Dyson-Schwinger gap equation. The $\times$ represents an insertion of the operator $\Sigma(k)$ everywhere except in diagram (a). In diagram (a) the $\times$ is $\Sigma(k)[1-2A(k)]$.

for the approximate correctness of the above numerical value, has been provided recently by lattice simulations.[6] It is the purpose of this paper to check directly the reliability of the $1/N$ expansion for critical behavior by estimating the contribution of the next-order terms in the expansion of the kernel. If they are reasonably small for $N \approx N_c$, additional evidence will have been provided that there is an N_c and that the $1/N$ expansion correctly captures the physics leading to the critical behavior.

To include all terms to order $1/N^2$ in the expansion of the kernel we consider the graphical expansion of the

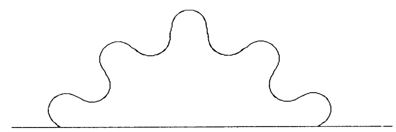

FIG. 2. First-order contribution to the wave-function renormalization $A(p)$. This term appears when computing the order-$(1/N^2)$ contribution of Fig. 1(a).

Dyson-Schwinger gap equation shown in Fig. 1. In order to study the gauge invariance of the theory, we make use of a ξ-dependent propagator which may be obtained from a nonlocal gauge-fixing term (note when $\xi=0$ we have the standard Landau-gauge photon propagator):[7]

$$D_{\mu\nu}(k)=\frac{g_{\mu\nu}-(1-\xi)(k)_\mu(k)_\nu/(k)^2}{(k)^2[1+\Pi(k)]}. \tag{7}$$

The gauge parameter ξ should drop out of the calculation of physical quantities. After checking this we will set $\xi=1$ to simplify calculations.

To begin, we first consider wave-function renormalization. From Fig. 2, we find (for $p/\alpha \ll 1$)

$$A(p)=4\frac{(2-3\xi)}{3\pi^2 N}\ln\left[\frac{p}{\alpha}\right]+\frac{c}{N}. \tag{8}$$

We next turn to the contributions to the gap equation. Figure 1(a) gives

$$\frac{2(2+\xi)\alpha}{\pi^2 Np}\int_0^\infty dk\,\frac{k\Sigma(k)}{k^2[1+A(k)]^2+\Sigma^2(k)}\ln\left[\frac{k+p+\alpha}{|k-p|+\alpha}\right]\approx\frac{4(2+\xi)}{\pi^2 N}\int_0^\alpha dk\frac{\Sigma(k)}{\max(k,p)}[1-2A(k)]. \tag{9}$$

We have used the same approximations as in Eq. (5) to linearize Eq. (9). In Ref. 5 the $A(k)$ term was ignored because it is of higher order in $1/N$. Here, we are including the $1/N^2$ terms. As seen in Eq. (8), $A(k)$ is of order $1/N$ and we expand the denominator of the left-hand side of Eq. (9) to include it.

The quantity we wish to consider is $\Sigma(p)/[1+A(p)]$ [it is this quantity, rather than $\Sigma(p)$, that leads to an N_c that is independent of ξ]. When we include all diagrams from Fig. 1 we find that Fig. 1(b) and the $1/N^2$ portion of Fig. 1(a) both contain logarithms of p/α [these are the second and third terms in the square brackets of Eq. (10a)]. The remaining diagrams, Figs. 1(c), 1(d), and 1(e), will contribute only to the constant piece in the kernel. We make use of the assumptions $k[1+A(k)] \gg \Sigma(k)$, and $\min(k,p)/\max(k,p) \ll 1$ in the integrand. This second approximation retains only the dominant behavior of the two-loop integrals. In particular, it keeps track of all possible ir and uv divergences. Adding all the diagrams in Fig. 1 we find

$$\frac{\Sigma(p)}{1+A(p)}=\frac{4(2+\xi)}{\pi^2 N}\int_0^\alpha dk\,\frac{\Sigma(k)}{\max(k,p)}\,\frac{1}{1+A(p)}\left[\frac{1+A(k)}{1+A(k)}\right]\left[1-2A(k)+\frac{8(2-3\xi)}{3\pi^2 N}\ln\left[\frac{\max(k,p)}{\alpha}\right]+\frac{c}{N}\right] \tag{10a}$$

$$=\frac{4(2+\xi)}{\pi^2 N}\int_0^\alpha\frac{dk}{\max(k,p)}\,\frac{\Sigma(k)}{1+A(k)}\left[1+\frac{4(2-3\xi)}{3\pi^2 N}\ln\left[\frac{\max(k,p)}{\min(k,p)}\right]+\frac{c}{N}\right]. \tag{10b}$$

It is important to note that the $\ln(p/\alpha)$ terms have canceled in the kernel. This cancellation is a direct consequence of the U(1) Ward identity. Were it not to occur, then the region where $\Sigma(0)$ is exponentially smaller than α (N near N_c) would lead to bad ir behavior in the integral, and this would qualitatively change our results. The potential danger of the $\ln(p/\alpha)$ terms, were they not to cancel, was noted in Ref. 8.

What remains can be seen to depend upon the anomalous dimension λ of the fermion field [see Eq. (8)]; this yields

$$\frac{\Sigma(p)}{1+A(p)}=\frac{4(2+\xi)}{\pi^2 N}\int_0^\alpha\frac{dk}{\max(k,p)}\left[\frac{\Sigma(k)}{1+A(k)}\right]\left[\frac{\max(k,p)}{\min(k,p)}\right]^\lambda\left[1-\frac{c}{\pi^2 N}\right], \tag{11}$$

where $\lambda = 4(2-3\xi)/3\pi^2 N$. Setting $\Sigma(p)/[1+A(p)] \propto p^b$, and momentarily considering only first order, we have

$$b(b+1) = -\frac{4(2+\xi)}{\pi^2 N} - \frac{4(2-3\xi)}{3\pi^2 N} = -\frac{32}{3\pi^2 N}. \quad (12)$$

Solving this gives $N_c = \frac{4}{3}(32/\pi^2)$ where all ξ dependence has dropped out. The difference between this N_c and the one found in Ref. 5 is the factor of $\frac{4}{3}$. This factor arises because we have included first-order contributions to the anomalous dimension λ. These contributions change the form of the kernel from that considered in Ref. 5. Higher-order terms in general should modify the anomalous dimension λ and the constant term c in Eq. (11), but should make no more qualitative changes to the kernel.

We have computed the second-order corrections with the gauge choice $\xi = 1$. The diagrams (d) and (e) of Fig. 1 were quite complicated and required numerical techniques to solve. The result is that in the equations that follow, all terms that are due to higher-order corrections to the vacuum polarization [such as Figs. 1(d) and 1(e)] depend on a numerical factor $a = 0.706$. All terms that are not proportional to a were found analytically and are exact.

In order to be consistent, we must find the $1/N^2$ contribution to the anomalous dimension λ. It is calculated to be

$$\lambda = -\frac{4}{3\pi^2 N} + \frac{134}{9\pi^4 N^2} - \frac{24a}{9\pi^4 N^2} = -\frac{4}{3\pi^2 N} + \frac{13.01}{\pi^4 N^2}. \quad (13)$$

The constant c from Eq. (11) is found to be

$$c = \frac{80+6a}{9} = 9.36. \quad (14)$$

Placing these values in Eq. (11) results in

$$b(b+1) = -\frac{32}{3\pi^2 N}\left[1 - \frac{341+48a}{48\pi^2 N}\right] = -\frac{32}{3\pi^2 N}\left[1 - \frac{7.81}{\pi^2 N}\right]. \quad (15)$$

Solving $b(b+1) = -\frac{1}{4}$, we find $N_c \approx 3.28$ (there are actually two solutions to this equation, but the second one gives an N_c near 1 and we consider it unphysical). This result shows that the second-order corrections are quite reasonable, changing N_c by only about 25%. More importantly, it shows that they do not qualitatively change our solution. We still have the same critical behavior as found in Ref. 5, and the new value of N_c is consistent with the results of the lattice calculations. While we have shown here that the second-order corrections are small, we have not proven that all higher-order terms may be neglected. However, we have presented evidence that, at the point of chiral-symmetry breaking, the critical value N_c is probably large enough that our expansion is reasonably convergent.

Our result demonstrates that an expansion of the kernel of the Dyson-Schwinger gap equation in powers of $1/N$ appears to describe reliably the critical behavior of the theory as a function of the number N of fermions.

I would like to thank T. Appelquist, T. Takeuchi, and M. Soldate for valuable discussions which were of great help in the preparation of this manuscript.

[1]T. Appelquist and R. Pisarski, Phys. Rev. D **23**, 2305 (1981).

[2]R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981).

[3]T. Appelquist and U. Heinz, Phys. Rev. D **24**, 2169 (1981).

[4]For a discussion of this question that does not consider the critical behavior of the fermion mass, see T. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. D **33**, 3774 (1986); S. Rao and R. Yahalom, California Institute of Technology Report No. CALT-68-1338 (unpublished).

[5]T. Appelquist, D. Nash, and L. C. R. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).

[6]E. Dagotto, A. Kocic, and J. Kogut, Phys. Rev. Lett. **62**, 1083 (1989).

[7]H. Nakatani (private communication).

[8]M. R. Pennington, University of Durham Report No. DTP 88/50 (to be published).