Do Steady Fast Magnetic Dynamos Exist?

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The question of whether it is possible to have a kinematic magnetic dynamo for a conducting fluid with time-independent (steady) velocity field $\mathbf{v}(\mathbf{x})$ and vanishingly small electrical resistivity has remained an open question in the understanding of the origin of magnetic fields in nature. By considering the zero-resistivity dynamics, examples of steady dynamos are found. Analysis of these examples supports the conjecture that, for sufficiently small resistivity, *dynamo action can occur in typical, smooth, steady, three-dimensional, chaotic fluid flows.*

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The kinematic magnetic dynamo problem[1-19] may be posed as follows: Given the flow of an initially unmagnetized electrically conducting fluid of velocity $\mathbf{v}(\mathbf{x},t)$, density $\rho(\mathbf{x},t)$, and electrical conductivity σ, will a small seed magnetic field amplify exponentially with time? If it does, then it is unnatural for the fluid to be unmagnetized, and it tends to generate its own magnetic field, i.e., the unmagnetized situation is unstable. Thus the kinematic dynamo problem is of interest with respect to the question of why magnetic fields are prevalent in astrophysical situations (e.g., planets, stars, galaxies, etc.). The basic (normalized) equation governing this situation is[1-4]

$$\partial(\mathbf{B}/\rho)/\partial t+\mathbf{v}\cdot\nabla(\mathbf{B}/\rho)=(\mathbf{B}/\rho)\cdot\nabla\mathbf{v}+(\rho R_m)^{-1}\nabla^2\mathbf{B}\,, \quad (1)$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ and ρ satisfy $\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{v})=0$, and R_m, the magnetic Reynolds number, is a normalized dimensionless electrical conductivity. Since R_m is extremely large in many situations (e.g., $R_m \gtrsim 10^8$ in the sun), it has been argued that only dynamos which survive in the limit $R_m \to \infty$ (called "fast" dynamos[6]) are of interest in these cases. It is currently an unresolved question as to whether fast kinematic dynamos can occur for typical smooth *steady* flows.[20] By a steady flow we mean one for which $\mathbf{v}$ is time independent [$\mathbf{v}=\mathbf{v}(\mathbf{x})$ and $\nabla\cdot(\rho\mathbf{v})=0$]. It is the question of the existence of steady fast kinematic magnetic dynamos which is the subject of this paper.[7-12,18-21] Our results suggest that typical, smooth, steady, three-dimensional, chaotic fluid flows can yield fast dynamos.

Here by "chaotic fluid flow" we mean that the trajectory of fluid elements, governed by the equation $d\mathbf{x}(t)/dt=\mathbf{v}(\mathbf{x}(t))$, displays a sensitive dependence on initial conditions. That is, if we consider two close-by orbits, $\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{x}}(t)+\delta\mathbf{x}(t)$ (where $\delta\mathbf{x}$ is infinitesimal), then, for typical $\hat{\mathbf{x}}(0)$ and $\delta\mathbf{x}(0)$, the quantity $|\delta\hat{\mathbf{x}}(t)|$ grows exponentially with time [$|\delta\mathbf{x}(t)|\sim\exp(ht)$, $h>0$]. The equation governing $\delta\mathbf{x}(t)$ is obtained from a variation of the fluid-element trajectory equation,

$$d\delta\mathbf{x}/dt=\delta\mathbf{x}\cdot\nabla\mathbf{v}(\hat{\mathbf{x}}(t))\,. \quad (2)$$

Arnold *et al.*[7] have considered the question of the existence of steady fast dynamos and introduced the idea that chaotic flows and fast dynamo action are connected. They illustrate this idea by explicitly constructing a steady fast dynamo based on a chaotic flow in a three-dimensional space of negative geodesic curvature. While this illustration is instructive, because of the special topological properties of the space required for their example, it is not clear what their example implies for flows in actual Euclidian space.

The relevance of chaotic flows follows by setting $R_m=\infty$ in (1) (i.e., deleting the $\nabla^2\mathbf{B}$ term). The equation for $\mathbf{b}\equiv\mathbf{B}/\rho$ then becomes the same as that for $\delta\mathbf{x}$, Eq. (2) ($d/dt\equiv\partial/\partial t+\mathbf{v}\cdot\nabla$). Hence exponential growth of $\delta\mathbf{x}$ appears to imply the possibility of exponential growth of $\mathbf{b}$ and hence $\mathbf{B}$. We note, however, that the limit $R_m\to\infty$ is highly singular,[9,15-17] and it is not immediately clear how the solution of (1) with $R_m=\infty$ relates to its solution with finite large R_m. An answer to this question was proposed in Refs. 15 and 16 and will be used here as a basis for our subsequent considerations in the present paper. Specifically, let $\Phi_\infty(t)$ denote the magnetic flux through some typical smooth surface where $\Phi_\infty(t)$ is obtained by solving (1) with $R_m=\infty$ and some typical smooth initial condition, $\mathbf{B}(\mathbf{x},0)$. Suppose Φ_∞ is observed to increase exponentially with time, $\Phi_\infty\sim\exp(\gamma_\infty t)$. Then, according to Refs. 15 and 16, the maximum unstable growth rate at finite R_m, $\gamma_{\max}(R_m)$, approaches γ_∞ as $R_m\to\infty$. (This statement has also recently been verified in numerical experiments.[17]) The importance of this is that it reduces the partial differential equation problem, Eq. (1) with finite R_m, to a problem in nonlinear dynamics, and in particular to $d\mathbf{x}/dt=\mathbf{v}$ and Eq. (2). Thus, in what follows, we shall attempt to specify a flow for which Φ_∞ increases exponentially with time. We call such flows D flows (where D stands for dynamo). If we construct a *steady* D flow, then we claim that we have a steady fast dynamo.

It is useful first to consider the case of a two-dimensional planar flow in which $\mathbf{B}$ lies in the same plane as the flow (i.e., $\mathbf{v}$ and $\mathbf{B}$ have no z component and no spatial dependence on z). In this case it is known that dynamo action cannot occur for any R_m (Cowling's theorem[1-4]). In the $R_m=\infty$ case this follows from the fact that magnetic field lines are convected with the fluid.[1-4] This is illustrated in Fig. 1, which shows an initial closed field line in Fig. 1(a) and the same field line after stretching by a two-dimensional flow [Fig. 1(b)]. The flux through the dashed line in the figure does not grow exponentially due to cancellation of upward and downward piercings of the dashed line by the magnetic field.

We now consider a class of three-dimensional steady flows which are periodic in z with periodicity length L. We shall also assume the field is periodic in z (although this is not necessary and Floquet solutions can also be readily analyzed[21]). For convenience of presentation we divide the interval $0\le z\le L$ into two regions. The first region occupies $0\le z\le \tilde{L}$ and is called the "map region." The second region, $\tilde{L}\le z\le L$, is called the "delay region." The z component of $\mathbf{v}$ is assumed to be positive everywhere, $v_z(x,y,z)>0$. In the map region the flow is z independent and incompressible, $\mathbf{v}(x,y,z)=v_0\mathbf{z}_0+\mathbf{v}_\perp(x,y)$, where $v_z\equiv v_0$ is constant and $\mathbf{v}_\perp$ is the transverse flow velocity having $\nabla\cdot\mathbf{v}_\perp=0$. Thus a fluid element entering the map region at a point $(x_i,y_i,0)$ exits it at a time $T_1\equiv\tilde{L}/v_0$ later at a position $(x_0,y_0,\tilde{L})$, and the input and output transverse positions are related to each other by a two-dimensional area-preserving map, $\mathbf{M}$, $(x_0,y_0)=\mathbf{M}(x_i,y_i)$. In the delay region, we take $\mathbf{v}_\perp\equiv 0$, but we let v_z depend smoothly on x, y, and z; $\mathbf{v}(x,y,z)=v_z(x,y,z)\mathbf{z}_0$ with $v_z(x,y,L)=v_z(x,y,\tilde{L})=v_0$. Thus in the delay region the transverse coordinates of fluid elements remain unchanged $(x,y)=(x_0,y_0)$, but elements exiting the map region take different amounts of time to traverse the delay region;[22] $T_2(x,y)=\int_{\tilde{L}}^{L}dz/v_z(x,y,z)$. In the special case where T_2 is independent of x and y, the inputs to the map region at time t all simultaneously return to the input to the map region at time $t+T$, where $T=T_1+T_2$. Thus the dynamics is represented by repeated application of $\mathbf{M}$, and the situation is essentially two dimensional. Hence, no dynamo action can occur. When $T(x,y)=T_1+T_2(x,y)$ depends on (x,y), this reasoning no longer holds, and one might suspect that there is a possibility that dynamo action could occur.

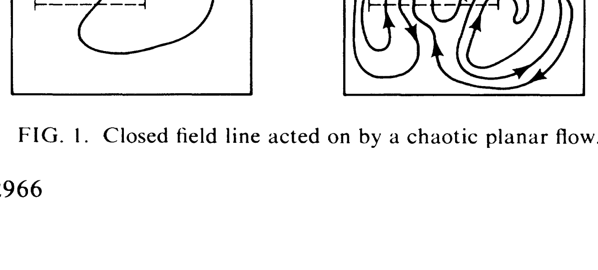

FIG. 1. Closed field line acted on by a chaotic planar flow.

We now consider a specific model $\mathbf{M}(x,y)$ and $T(x,y)$ for which we can obtain analytical results. It is assumed that the fluid is of unbounded extent in (x,y). The flow in the map region is specified geometrically in Fig. 2. The flow takes the two-dimensional, square region, $S=\{x,y\,|\,0\le x\le 1,\ 0\le y\le 1\}$, at the input plane, $z=0$, and deforms it into the shape shown in Fig. 2(c) at the output plane, $x=\tilde{L}$. Thus $\mathbf{M}(x,y)$ is a "horseshoe map." We emphasize that horseshoes of this type can be expected to arise naturally and that the flow $\mathbf{v}_\perp$ which gives $\mathbf{M}$ can be taken to be continuous and as smooth as we like. We specify $T(x,y)$ as

$$T(x,y)=\begin{cases}T_a & \text{in region } a\,,\\ T_b & \text{in region } b\,,\end{cases} \tag{3}$$

where T_a and T_b are constants, and a and b label the shaded regions shown in Fig. 2. For our example it is not necessary to specify the form of the function $T(x,y)$ outside regions a and b, but we can (and do) require $T(x,y)$ to be smoothly varying everywhere. Thus our flow is smooth (no discontinuities or singularities as occur in other previous work[20]). We imagine that at $t=0$ the magnetic field through S is directed upward (as indicated in the figure) and that the field lines close on each other by circling a finite distance to the left of S. Thus all the initial magnetic field is contained in a finite region in (x,y), and the initial magnetic energy per unit length in z is finite. For our subsequent discussion, we will neglect the effect of the initial field circling to the

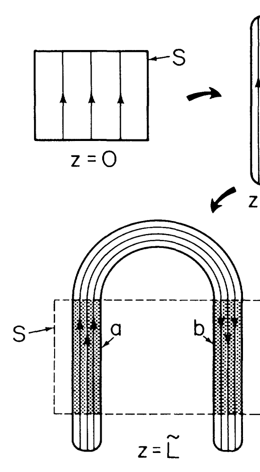

FIG. 2. Deformation of the square S with frozen-in magnetic field lines. Three cross sections corresponding to three values of z are shown $(0<z_1<\tilde{L})$.

left of S, since it may be shown[23] that it does not effect our considerations, to follow, in determining the exponential flux growth rate. Consider a surface $y=\frac{1}{2}$, $0 \le x \le 1$, and denote the total flux through this surface per unit length in z by $\Phi(z,t)$ (we take Φ as positive if the flux is in the positive y direction). Since (1) is a time-independent linear equation, we look for eigensolutions $\Phi(z,t)=\phi(z)\exp(st)$, $s=i\omega+\gamma$. Now say we consider the flux per unit length at time t at the output to the map region, $z=\tilde{L}$. From Fig. 2 and Eq. (3), we see that at time t the upward flux per unit length at $z=\tilde{L}$ in region a is the same as the total upward flux per unit length which existed at $z=0$ at the time $t-T_1$. Similarly, at time t the downward flux per unit length at $z=\tilde{L}$ in region b is equal to the flux per unit length at $z=0$ at the time $t-T_1$. These fluxes then flow through the delay region eventually again arriving at the input to the map region. Thus $\Phi(0,t)=\Phi(0,t-T_a)-\Phi(0,t-T_b)$. Inserting $\Phi(0,t) \sim \exp(st)$ yields

$$\lambda^{\rho}-\lambda^{\rho-1}+1=0, \tag{4}$$

where $\rho=T_b/T_a$ and $\lambda=\exp(sT_a)$. Roots of (4) with $|\lambda|>1$ [$\mathrm{Re}(s)=\gamma>0$] imply exponential flux growth (i.e., the flow is a steady D flow).

We now investigate the solutions of Eq. (4). Writing λ as $\lambda=re^{i\theta}$ and (4) as $1-\lambda=\lambda^{-(\rho-1)}$ and taking the magnitude of both sides of this equation, we find that the roots of Eq. (4) must lie on the following polar coordinate curve in the λ plane:

$$\cos\theta=\tfrac{1}{2}\left[r+r^{-1}-r^{-(2\rho-1)}\right]. \tag{5}$$

This curve is shown schematically in Fig. 3, where r_+ denotes the positive real root of $r_+^{\rho}-r_+^{\rho-1}-1=0$ and r_- denotes the positive real root of $r_-^{\rho}+(r_-^{\rho-1}-1)\times\mathrm{sgn}(\rho-1)=0$. Note that $r_+>1$ and $r_-<1$ for all ρ. For integer or rational values of ρ, Eq. (4) has a finite number of roots which lie on the curve given by (5). For example, for $\rho=2$, $\lambda=\frac{1}{2}(1\pm i\sqrt{3})$ which yields stability, $|\lambda|=1$. For $\rho=1$, the only root is $\lambda=0$, which again yields stability. For integer values of ρ larger than 2 we always find instability (i.e., roots with $|\lambda|>1$). More importantly, however, for irrational ρ, Eq. (4) has a discrete infinity of roots which densely fill the curve (5). Thus there are roots with $|\lambda|=e^{\gamma T_a}$ arbitrarily close to $r_+>1$. Hence, if ρ is chosen at random, we have a D flow with probability 1.

It is important to note that, while the roots in λ are typically dense on a curve [Eq. (5)], in the s plane they are isolated. It is instructive to consider an example. Say $\rho=1+\epsilon$ with $0<\epsilon\ll 1$. For ϵ small and positive, the curve in Fig. 3 is close to a circle of radius 1 centered at $\mathrm{Re}(\lambda)=1$, $\mathrm{Im}(\lambda)=0$. The curve passes slightly to the left of $\lambda=0$ ($r_- \sim \epsilon\ln\epsilon^{-1}$) for $\epsilon>0$. The curve in the λ plane then translates via $\lambda=e^{sT_a}$ to the solid curve in the s plane shown schematically in Fig. 4. To find a second curve on which the roots also must lie, we write (4) as $1-\lambda^{\epsilon}=-\lambda^{1+\epsilon}$, take magnitudes of both sides, and utilize $\epsilon\ll 1$. We obtain

$$\gamma T_a \cong \ln 2+\ln|\sin(\epsilon\omega T_a/2)|, \tag{6}$$

which is plotted as the dashed curve in Fig. 4. Any roots must lie on the intersections of the two curves. In fact, further examination of Eq. (4) shows that only every other intersection (labeled as dots in Fig. 4) yields an actual root. Thus, for our example, even the slightest deviation ($\epsilon\neq 0$) from the two-dimensional case ($\epsilon=0$) typically produces a steady D flow. Note from (6) that unstable roots ($\gamma>0$) are possible only if their imaginary parts exceed a critical value $\omega>\omega_c\cong\pi/3\epsilon T_a$. {Thus for very small ϵ the unstable roots must have very large ω. This leads to rapid oscillations in z [in the delay region the fields vary as $\exp(-i\omega\int^z dz/v_z)$], with the consequence that for small ϵ the unstable modes are severely effected by relatively small resistive field diffusion.}

The basic mechanism leading to the D flow in our example is similar to that leading to dynamo action in the *nonsteady* flow of Ref. 14 and is as follows. For $T(x,y)$ =const, perfect cancellation of flux occurs (Fig. 1). When $T(x,y)$ is not constant the existence of a nonzero frequency of oscillation (imaginary part of s) leads to phase shifts which alter the magnetic field at different

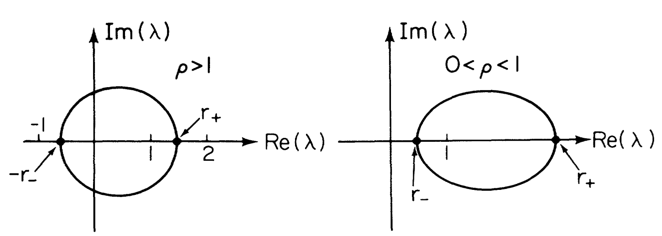

FIG. 3. The curve (5) in the λ plane.

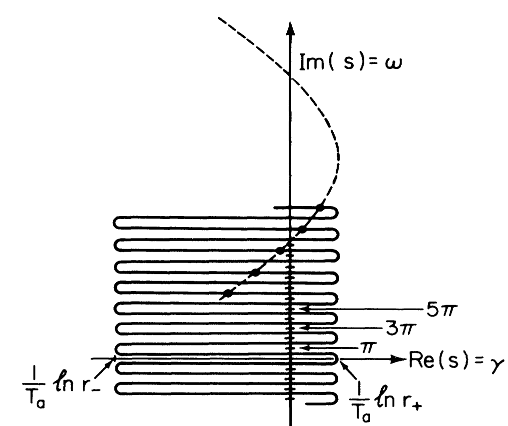

FIG. 4. Roots in the s plane (dots) for ρ slightly larger than 1.

(x,y) positions by a factor $\exp[-i\omega T(x,y)]$. These phase shifts spoil the perfect cancellation which occurs in the two-dimensional case and can lead to flux growth. For example, for Eq. (3) with $\rho=1+\epsilon$ and $\epsilon \ll 1$, a phase shift between the strips of π (i.e., $\epsilon\omega T_a=\pi$) changes the exact two-dimensional cancellation to in-phase addition, and the flux growth rate is $\gamma=\ln 2$ [cf. Eq. (6)]. While it will not be possible to obtain precise in-phase behavior at all (x,y) points for general $\mathbf{M}(x,y)$ and $T(x,y)$, the basic effect (i.e., mitigation of exact cancellation) will still be present, and our example indicates that it can be strong enough to lead to growth. Thus we expect that the D-flow property is common for steady, smooth, three-dimensional, chaotic fluid flows, and fast dynamos should consequently be a typical occurrence for such flows.

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[22]From $\nabla\cdot(\rho\mathbf{v})=0$, the fluid density in the delay region is $\rho(x,y,z)=\rho_0 v_0/v_z(x,y,z)$, and $\rho=\rho_0$ in the map region, where ρ_0 is a constant. If desired we can make the flow incompressible everywhere by smoothly deforming the flow lines in the delay region while leaving v_z along a flow line [and hence $T(x,y)$] unchanged. In particular, we introduce functions $\tilde{x}=\tilde{x}(x_0,y_0,z)$, $\tilde{y}=\tilde{y}(x_0,y_0,z)$ which specify the transverse coordinates of the deformed flow lines labeled by the position of the flow line (x_0,y_0) at $z=\tilde{L},L$. These functions are chosen so that the Jacobian $\partial(\tilde{x},\tilde{y})/\partial(x_0,y_0)$ is $\rho(x_0,y_0,z)/\rho_0$. Corresponding to the flow line deformation, there is now a transverse velocity,

$$\tilde{\mathbf{v}}_\perp(\tilde{x},\tilde{y},z)=v_z(x_0,y_0,z)[\mathbf{x}_0\partial\tilde{x}/\partial z+\mathbf{y}_0\partial\tilde{y}/\partial z].$$

For example, one of the infinite number of possible choices which has the required Jacobian is $\tilde{x}=x$, $\tilde{y}=\int_0^y(\rho/\rho_0)dy$.

[23]J. M. Finn, J. D. Hanson, I. Kan, and E. Ott (to be published). In this paper we will also report on numerical computations for more general flows and dependences of $\mathbf{M}$ and T.