Universality of Cubic-Level Repulsion for Dissipative Quantum Chaos

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The quantum signature of chaos in dissipative systems is cubic repulsion of their generalized energies (eigenvalues of the generators of the dynamics) in the complex plane. As in the Hamiltonian case the degree of repulsion is the same under temporally homogeneous conditions and periodic driving. Somewhat surprisingly, however, cubic repulsion prevails irrespective of whether the Hamiltonian embedding of the dissipative system obeys time-reversal invariance. Even antiunitary symmetries of the dissipative generator itself cannot modify the repulsion exponent. In establishing the universality in question we find the generalization of detailed balance for periodically driven damped systems.

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As is well known, Hermitian Hamiltonians H as well as unitary Floquet operators F with a nonintegrable classical limit require two, three, or five controllable parameters for an eigenvalue crossing to become a generic possibility rather than an unlikely exception.[1-4] The number n in question is determined by the set of unitary and antiunitary symmetries of H or F. Certain universal features of the spectra of eigenvalues of H and eigenphases (quasienergies) of F are distinct for each of the three classes of Hamiltonians and Floquet operators. Most notably, the degree of repulsion of neighboring energies or quasienergies is given by $n-1$.

A fourth universality class of Hermitian and unitary operators comprises the cases with integrable classical limits. These have $n=1$; more specifically, their levels typically have a tendency to cluster just as if arising as events in a Poissonian random process.

Matrices not restricted by Hermiticity or unitarity have recently found applications as generators of dissipative quantum maps.[5-7] Such maps arise for periodically driven systems with damping, as stroboscopic period-to-period descriptions of the time evolution of the density matrix. The eigenvalues of the corresponding matrices are in general complex and, for stable systems, smaller than or at most equal to unity in modulus. In the limit of zero damping when the generator becomes unitary the eigenvalues all lie on the circumference of the unit circle around the origin. As the damping is increased, however, the eigenvalues wander towards the origin; expected from that tendency is, in general, only one eigenvalue at unity which pertains to the stationary density matrix of the map.

Defining the spacing S between two eigenvalues of a general matrix as their Euclidean distance in the complex plane, for each eigenvalue a nearest neighbor can be found and the distribution $P(S)$ of nearest-neighbor spacings can be established.

By numerically diagonalizing the generators for a class of periodically driven damped spins the spacing distribution was recently shown[7] to display linear and cubic "level" repulsion when the classical motion is predominantly regular and chaotic, respectively,

$$P(S) \sim \begin{cases} S, & \text{regular,} \\ S^3, & \text{chaotic,} \end{cases} \tag{1}$$

for $S \rightarrow 0$. In fact, the respective distributions $P(S)$ turned out to be well described by those of a Poissonian random process in the plane and of Ginibre's ensemble of general complex random matrices.[8]

An interesting question now arises as to whether the generators of classically nonintegrable dissipative quantum dynamics can fall into different universality classes with different degrees of level repulsion. A stimulus to investigate that question may be seen in the fact that any dissipative motion can be understood as a subdynamics of the Hamiltonian motion of the system in consideration coupled to a heat bath. The corresponding total Hamiltonian may or may not have antiunitary symmetries (time reversal, conventional or generalized) and may therefore, if classically nonintegrable, display any of the universal degrees of level repulsion mentioned above. Moreover, there is at least one important property by which a dissipative motion reveals whether or not the underlying microscopic dynamics is time-reversal invariant: Detailed balance holds if and only if the embedding has a Hamiltonian with an antiunitary symmetry.[9-11]

A bit of numerical evidence for the cubic repulsion of complex levels to be robust against changes of antiunitary symmetries of the microscopic embedding was presented in Ref. 7. Here we propose to demonstrate that no other universality classes of nonintegrable dissipative maps exist, at least as far as the degree of level repulsion is concerned. Our argument is based on almost degenerate perturbation theory and thus is similar to the one used to find the well-known universality classes of Hermitian and unitary matrices.[4,12]

Imagine two complex eigenvalues of a matrix D to undergo a close encounter when a parameter in D is varied. We may assume that the corresponding two eigenvectors are known for a particular value of the parameter in question. By diagonalizing D in that two-dimensional space the fate of the eigenvalues throughout the encounter is found. Especially, the difference of the two eigenvalues arises as

$$D_+ - D_- = \{(D_{11}-D_{22})^2+4D_{12}D_{21}\}^{1/2}. \qquad (2)$$

The spacing distribution for a matrix of large dimension is then accessible as

$$P(S) = \langle \delta(S - |D_+ - D_-|) \rangle , \qquad (3)$$

the average being over all close encounters in the spectrum of D or, formally speaking, over the matrix elements D_{ij} with suitable distributions $W_{ij}(D_{ij})$ for each of them. Since we inquire about the behavior of $P(S)$ for $S \to 0$ we must avail ourselves of the W_{ij} for $D_{ij} \to 0$ only. The crucial property yielding universal behavior of $P(S)$ is

$$W_{ij}(D_{ij}) \underset{D_{ij}\to 0}{\longrightarrow} \text{const} \; (\neq 0, \infty) . \qquad (4)$$

In the case of unitary or Hermitian matrices D the integrals in (3) need not be evaluated explicitely to find $P(S)$ for small S. Rather, by simply rescaling the integration variables as $D_{ij} \to SD_{ij}$, using (4), and realizing the number of non-negative terms in the discriminant in (2) to be just n, one finds the power law[13] $P(S) \sim S^{n-1}$.

When dealing with general matrices, however, the discriminant in (2) is not a sum of non-negative terms. The distribution $P(S)$ must therefore be determined by actually integrating over the four D_{ij} in (3). That elementary task involves the following four types of complex integrals: (i)

$$\int d^2x\, d^2y\, W(x)W(y)\delta^2(z-xy) \sim \ln|z| ,$$

(ii)

$$\int d^2x\, W(x)\delta^2(z-x^2) \sim 1/|z| ,$$

(iii)

$$\int d^2x\, d^2y \frac{W(y)}{|x|} \ln|y|\, \delta^2(z-x-y) \sim \text{const} \neq 0 ,$$

and (iv)

$$\int d^2x\, W(x)\delta^2(S-|\sqrt{x}|) \sim S^3 .$$

These auxiliary integrals are all meant in the limit of small $|z|$ and S; the $W(x)$ are assumed to obey (4). We thus find

$$P(S) \sim S^3 \text{ for } S \to 0 . \qquad (5)$$

In deriving (5) we have assumed the elements of D_{ij} unrestricted by any symmetries. That assumption is justified when the Hamiltonian embedding has no generalized time reversal and thus displays quadratic repulsion of adjacent energy levels. We now proceed to show that the D_{ij} remain effectively unrestricted even if the embedding does have an antiunitary symmetry. To that end we must be a little more specific about the embedding $\mathcal{S}+\mathcal{R}$ which consists of the system $\mathcal{S}$ under consideration and a reservoir $\mathcal{R}$. For the sake of definiteness we consider periodically driven systems for which a stroboscopic period-to-period description involves a unitary Floquet operator F. The density operator W of $\mathcal{S}+\mathcal{R}$ obeys the unitary quantum map $W_{n+1}=FW_nF^\dagger$ in which the discrete time $n=0,1,2,\ldots$ counts the number of periods passed. We are interested in a stationary regime corresponding to a density operator commuting with F, $\overline{W}=F\overline{W}F^\dagger$. The assumed (generalized) time-reversal invariance of the dynamics of $\mathcal{S}+\mathcal{R}$ is meant as implying[4] $TFT^{-1}=F^\dagger$ and $T\overline{W}T^{-1}=\overline{W}$, where T is some antiunitary operator.

For our purposes we need correlation functions of two (not necessarily Hermitian) observables A and B, $\langle A(n)B\rangle = \mathrm{tr}_{\mathcal{S}\mathcal{R}} A(n)B\overline{W}$, where $A(n)=F^{\dagger n}AF^n$. Conventional arguments[10,11] can easily be extended to our discrete-time dynamics to show that the assumed time-reversal invariance entails the identity[14]

$$\langle A(n)B\rangle = \langle \tilde{B}(n)\tilde{A}\rangle, \quad \tilde{A}=TA^\dagger T^{-1} \equiv \tau A . \qquad (6)$$

We intend to employ the foregoing identity for the case where A and B are observables of $\mathcal{S}$ alone, i.e., behave like unity with respect to $\mathcal{R}$. Both correlation functions in (6) may then be calculated using the dissipative subdynamics of $\mathcal{S}$ alone, which in the situation of interest is described by a dissipative map for the reduced density operator $\rho_n = \mathrm{tr}_{\mathcal{R}} W_n$ of $\mathcal{S}$,

$$\rho_{n+1} = D\rho_n . \qquad (7)$$

The correlation function $\langle A(n)B\rangle$ can be written with the help of the generator D as[15]

$$\langle A(n)B\rangle = \mathrm{tr}_{\mathcal{S}} AD^n(B\bar{\rho}) = \mathrm{tr}_{\mathcal{S}} B\bar{\rho}D^{\dagger n}A . \qquad (8)$$

In order to similarly express $\langle \tilde{B}(n)\tilde{A}\rangle$ we must realize that the antiunitary operator T can be given a meaning with respect to observables of $\mathcal{S}$. For $\mathcal{S}+\mathcal{R}$ there is no loss of generality in writing T as the product of the complex-conjugation operation C and two unitary operators $Y_{\mathcal{S}}$ and $Y_{\mathcal{R}}$ which refer, respectively, to $\mathcal{S}$ and $\mathcal{R}$, $T=Y_{\mathcal{R}}Y_{\mathcal{S}}C$. It is then easy to see that $\tilde{A}=\tau A = Y_{\mathcal{S}}CA^\dagger CY_{\mathcal{S}}^{-1} = \tau_{\mathcal{S}}A$. With $T_{\mathcal{S}}=Y_{\mathcal{S}}C$ as the representative of the time-reversal operation at our disposal we can read and calculate both correlation functions in the identity (6) without further reference to the heat bath $\mathcal{R}$. Especially, by using the $T_{\mathcal{S}}$ invariance of the stationary reduced density operator $\bar{\rho}$ and assuming the existence of $\bar{\rho}^{-1}$, we have

$$\langle \tilde{B}(n)\tilde{A}\rangle = \mathrm{tr}_{\mathcal{S}}(\tau_{\mathcal{S}}B)D^n((\tau_{\mathcal{S}}A)\bar{\rho})$$
$$= \mathrm{tr}_{\mathcal{S}} B\bar{\rho}(\tau_{\mathcal{S}}\bar{\rho})^{-1}D^n(\tau_{\mathcal{S}}\bar{\rho}A) . \qquad (9)$$

The inverse of the linear tetradic operator $\tau_{\mathcal{S}}\bar{\rho}$ is defined to act on arbitrary observables X of $\mathcal{S}$ as

$$(\tau_{\mathcal{S}}\bar{\rho})^{-1}X=\bar{\rho}^{-1}\tau_{\mathcal{S}}^{-1}X=\bar{\rho}^{-1}T_{\mathcal{S}}^{-1}X^{\dagger}T_{\mathcal{S}}.$$

The identity (6) holds for arbitrary observables A and B of $\mathcal{S}$ and therefore implies an important property of the generator D. Indeed, by comparing (7) and (8) we find

$$(\tau_{\mathcal{S}}\bar{\rho})^{-1}D\tau_{\mathcal{S}}\bar{\rho}=D^{\dagger}, \tag{10}$$

a property identical in appearance with detailed balance for the generator of infinitesimal time translations of a dissipative system with a time-reversal-invariant autonomous Hamiltonian embedding.[10] We have, in fact, derived the analog of detailed balance for the stroboscopic evolution of periodically driven dissipative systems.

Since the map (7) must conserve the Hermiticity of the density operator of $\mathcal{S}$, the generator D must obey $(DA)^{\dagger}=DA^{\dagger}$ for any non-Hermitian operator A. It is easy to see that the latter property, together with the condition (10) for detailed balance, implies that a right-hand eigenoperator A of D is accompanied by a left-hand eigenoperator

$$A^{L}=(\tau\bar{\rho})^{-1}A^{\dagger} \tag{11}$$

pertaining to the same eigenvalue λ. The pairs of left- and right-hand eigenoperators form a biorthogonal system, $\langle A_i^L|A_j\rangle=\delta_{ij}$.

The 2×2 matrix describing a close encounter of two eigenvalues $D_{\pm}$ may now be written as $D_{ij}=\langle A_i^L|DA_j\rangle$, where A_i and A_i^L are two pairs of left- and right-hand eigenvectors of a generator $D^{(0)}$ infinitesimally close to D with respect to some control parameter of the dissipative dynamics. Detailed balance is assumed to hold for both D and $D^{(0)}$; it is important to realize, though, that the condition (10) involves the, in general, different stationary eigenoperators $\bar{\rho}$ and $\bar{\rho}^{(0)}$ of D and $D^{(0)}$. The diagonal elements D_{11} and D_{22} cannot be expected to be related to one another by detailed balance since even in the zero-damping limit they are left independent by time-reversal invariance. Less obvious is the fact, to be revealed presently, that the symmetry $D_{12}=D_{21}$ imposed by time-reversal invariance in the zero-damping limit breaks down for finite damping. But indeed, we may rewrite $D_{12}=\langle A_1^L|DA_2\rangle$ by using (11) to express A_1^L and A_2 in terms of $\bar{\rho}_0$ and, respectively, A_1 and A_2^L; we then employ detailed balancing for D according to (10) and use the definition of the adjoint tetrad $D^{\dagger}$ with respect to the scalar product of observables to arrive at

$$\langle A_1^L|DA_2\rangle=\langle\bar{\rho}_0A_2^L\bar{\rho}^{-1}|D\bar{\rho}_0^{-1}A_1\bar{\rho}\rangle. \tag{12}$$

While this latter identity is easily seen to reduce to the symmetry $D_{12}=D_{21}$ in the zero-damping limit, it does not imply any relation between the two matrix elements D_{12} and D_{21} alone in the dissipative case. Rather, if we imagine $\bar{\rho}_0^{-1}A_1\bar{\rho}$ expanded in terms of the complete set of right-hand eigenoperators of $D^{(0)}$ and, similarly, $\bar{\rho}_0A_2^L\bar{\rho}^{-1}$ in terms of the corresponding left-hand eigenoperators, we see that (12) relates D_{12} linearly to all the other matrix elements of D; such a "global" relation does not, of course, restrict the two-by-two matrix describing the near miss of two eigenvalues of D. We must conclude that for finite damping the repulsion of two eigenvalues is insensitive to whether or not D obeys detailed balance, i.e., to whether or not the embedding of the dissipative system into a larger Hamiltonian one is time-reversal invariant.

Incidentally, if the generator D itself has a covariance[16] $ADA^{-1}=D^{\dagger}$, with A antiunitary and $A^2=+1$, of whatever physical origin, conventional arguments[1-4] would reveal the symmetry $D_{12}=D_{21}$. Still, since in the dissipative case D is neither Hermitian nor unitary, the degree of level repulsion would remain cubic to within only a logarithmic correction. Indeed, by repeating the elementary integration leading to (5) we now find $P(S)\sim S^3\ln S$ for $S\rightarrow 0$. Similarly, if D displayed a Kramers kind of degeneracy (due to $ADA^{-1}=D^{\dagger}$, with A antiunitary and $A^2=-1$), the pairwise-degenerate eigenvalues would also repel cubically. As was shown in Ref. 4 the close encounter of two eigenvalues must then be described by a 4×4 submatrix of D. The corresponding spacing of eigenvalues $|D_+-D_-|$ leads to $P(S)\sim S^3$ as is readily checked by again going through the integration routine described above.

We would like to conclude by pointing out that all of the above results hold unchanged when D is the generator of infinitesimal time translation of some master equation rather than of discrete-time quantum maps. Cubic level repulsion thus appears as a universal property of dissipative quantum dynamics with a chaotic classical limit.[5,17]

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[13]To treat the case with $n=5$ one must secure Kramers degeneracy by working with properly restricted 4×4 matrices.

[14]Note that $\tilde{A}$ is linearly (rather than antilinearly) related to A.

[15]The adjoint $D^\dagger$ of the (tedradic) generator D is defined with respect to the scalar product $\langle X|Y\rangle = \mathrm{tr} X^\dagger Y$ of the observables X and Y. The adjoint $X^\dagger$ of the observable X, on the other hand, refers to the usual scalar product of wave vectors.

[16]Hermiticity conservation implies $ADA^{-1}=D$, with A antiunitary and $A^2=1$. This symmetry does not imply a restriction for the 2×2 matrix either.

[17]An exception to that rule might be constituted by systems displaying quantum localization [cf. S. Fishman, D. R. Grempel, and R. P. Prange, Phys. Rev. Lett. **49**, 509 (1982)].