Heavy Top-Quark Mass Predictions

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Renormalization-group equations and asymptotic conditions are used to obtain top-quark mass predictions. In the standard model with an explicit Higgs mechanism, our analysis becomes identical to the Kubo-Sibold-Zimmermann coupling-reduction approach. Refining their prediction, we find $m_t \simeq 95$ GeV. In the simplest case of self-consistent dynamical symmetry breaking, we obtain $m_t \simeq 115$ GeV. An illustration of how short-distance new physics can alter these predictions is given.

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It now appears that the top quark is much more massive than originally anticipated. Direct searches at KEK, CERN, and Fermilab have, so far, failed to find the top quark. Instead, they provide[1] the lower bound $m_t > 44$ GeV. On a separate front, the observed[2] large B_d^0-$\bar{B}_d^0$ oscillation rate seems to suggest $m_t \gtrsim 55$ GeV, with most theoretical analyses favoring higher values. At the other extreme, electroweak phenomenology including top-quark loop effects gives[3] $m_t \lesssim 200$ GeV. So, even though the top quark has not been directly detected, it is rather safe to assume that its mass lies in the range

$$44\ \text{GeV} \lesssim m_t \lesssim 200\ \text{GeV}. \tag{1}$$

The apparent large value of the top-quark mass is in sharp contrast with the masses of all other known fermions. In fact, it appears that m_t is much closer to the $W^\pm$- and Z-boson masses

$$m_W = 81 \pm 2\ \text{GeV}, \qquad m_Z = 92 \pm 2\ \text{GeV}, \tag{2}$$

than to its fermionic brethren. Given that situation, it is natural to ask whether there might be some calculable relationship between m_t and m_W that would allow one to predict the top-quark mass before its discovery.

Here we would like to present predictions for m_t that are based on renormalization-group equations and assumptions regarding the asymptotic behavior of their solutions. In the first case, we consider the standard $SU(3)_C \times SU(2)_L \times U(1)$ model with an explicit Higgs mechanism. There, although our starting point is somewhat different, the analysis becomes identical to results by Kubo, Sibold, and Zimmermann (KSZ).[4] Those authors employed a coupling-reduction technique[5] in which Yukawa couplings and Higgs-boson self-couplings were expressed as a function of the $SU(3)_C$ gauge coupling α_3 such that renormalizability was preserved. They found in lowest order[4]

$$m_t^2 \simeq \frac{4}{9}\frac{\alpha_3(m_W)}{\alpha_2(m_W)} m_W^2, \tag{3}$$

with α_3 and α_2 the $SU(3)_C$ and $SU(2)_L$ gauge couplings. After including electroweak effects, KSZ gave the numerical prediction $m_t \simeq 81$ GeV, $m_{\text{Higgs}} \simeq 61$ GeV, with quoted uncertainties of about (10–15)%. The same lowest-order prediction in Eq. (3) was also obtained by Pendleton and Ross[6] (PR) in their analysis of infrared fixed points in grand unified theories. They used a larger $\alpha_3(m_W)$ as input and gave a numerical prediction[7] of $m_t \simeq 135$ GeV, $m_{\text{Higgs}} \simeq 72$ GeV. (Our subsequent refinement of the KSZ analysis gives $m_t \simeq 95$ GeV.)

In our second scenario, a dynamical symmetry-breaking framework, i.e., no explicit Higgs scalar, is assumed. Fermion and weak-gauge-boson masses are included in loop effects self-consistently in the spirit of Nambu-Jona-Lasinio,[8] but required to vanish asymptotically where the symmetry of the massless theory is presumably restored. We find in lowest order

$$m_t^2 \simeq \frac{2}{3}\frac{\alpha_3(m_W)}{\alpha_2(m_W)} m_W^2 \ \text{(no Higgs boson)}, \tag{4}$$

which (after including electroweak and higher-order QCD) results in a prediction of $m_t \simeq 115$ GeV. The finding in (4) is reminiscent of general results obtained by Jackiw and Johnson[9] and Cornwall and Norton[10] in their classic papers on dynamical symmetry breaking. It is also in keeping with work by Carter and Pagels[11] which suggested that discovery of a heavy fermion near the $W^\pm$ or Z masses would be evidence in favor of dynamical symmetry breaking.

Both cases assume no new physics beyond the three-generation standard model. They are, however, quite sensitive to new short-distance effects which could increase m_t to the 200–250-GeV range or, if fourth-generation fermions exist, lower it somewhat.

We begin with the usual assumption that the local $SU(3)_C \times SU(2)_L \times U(1)$ gauge symmetry of the standard model is spontaneously broken to $SU(3)_C \times U(1)_{\text{em}}$ at a scale of $v \simeq 250$ GeV. That breaking gives rise to gauge-boson masses as well as fermion masses. The scale v may correspond to a Higgs-scalar vacuum expectation value or be related to the scale of fermion-antifermion vacuum condensation in dynamical scenarios, analogous to the BCS theory of superconductivity.

The specifics of the symmetry-breaking mechanism do not initially concern us. We need only assume that the $SU(3)_C \times SU(2)_L \times U(1)$ model with three generations of fermions (with or without a Higgs scalar) perturbatively describes all strong and electroweak physics. (Of course, the scale v itself is very likely to be of nonperturbative origin.) The $W^{\pm}$- and Z-boson masses are given by

$$m_Z^2 \cos^2\theta_W = m_W^2 = \pi\alpha_2(m_W) v^2 , \tag{5}$$

with $\sin^2\theta_W \simeq 0.23$ and $\alpha_2(m_W)$ the running $SU(2)_L$ gauge coupling evaluated at scale $\mu = m_W$. The fermion masses are related to v by

$$m_f^2 = 2\pi\kappa_f v^2 , \tag{6}$$

with κ_f arbitrary parameters in the standard Higgs mechanism (related to Yukawa couplings), generally fixed to accommodate phenomenology. [The κ_f are subsequently taken to be running couplings $\kappa_f(\mu)$ with $\kappa_f \simeq \kappa(m_f)$ up to calculable corrections.] Our goal is to compute the κ_f in terms of the known gauge couplings. Then, using

$$\kappa_f = \frac{\alpha_2(m_W)}{2}\frac{m_f^2}{m_W^2} = \frac{G_F}{\pi\sqrt{2}} m_f^2 \tag{7}$$

with $G_F = 1.166\,37 \times 10^{-5}$ GeV^{-2}, one can predict m_f. The gauge couplings $\alpha_i \equiv g_i^2/4\pi$, $i = 1,2,3$ are well determined[3,12] and relatively small at a scale $\mu = m_W$ [using couplings defined by modified minimal subtraction ($\overline{\text{MS}}$)],

$$\begin{aligned} \alpha_3(m_W) &= 0.107^{+0.010}_{-0.007} , \\ \alpha_2(m_W) &= 0.0344 \pm 0.0007 , \\ \alpha_1(m_W) &= 0.0169 \pm 0.0001 , \end{aligned} \tag{8}$$

so perturbation theory should be valid. At higher energies, their evolution is governed by[13]

$$\begin{aligned} \beta_3 &\equiv \mu\frac{d}{d\mu}\alpha_3(\mu) = \frac{-1}{2\pi}(11 - \tfrac{4}{3}n_g)\alpha_3^2 + \cdots , \\ \beta_2 &\equiv \mu\frac{d}{d\mu}\alpha_2(\mu) = \frac{-1}{2\pi}(\tfrac{22}{3} - \tfrac{4}{3}n_g - \tfrac{1}{6}N_H)\alpha_2^2 + \cdots , \\ \beta_1 &\equiv \mu\frac{d}{d\mu}\alpha_1(\mu) = \frac{-1}{2\pi}(-\tfrac{4}{3}n_g - \tfrac{1}{10}N_H)\alpha_1^2 + \cdots , \end{aligned} \tag{9}$$

where $\cdots$ represent higher orders, $n_g = 3$ generations (for $\mu \geq m_t$), and N_H is the number of explicit Higgs doublets. ($N_H = 1$ and 0 for the two cases we consider.) If "new" physics enters at some high-energy scale, (9) will be modified above that scale.

Since in the massless theory, only the gauge couplings α_i exist as independent parameters, we *assume* that the κ_f are functions of the α_i. The approach to small coupling can be studied for the $SU(3)_C$ coupling $\alpha_3(\mu)$ by going to large μ, since it is asymptotically free. In the case of $\alpha_1(\mu)$, it increases asymptotically (albeit slowly) while $\alpha_2(\mu)$ decreases but at a slower rate than $\alpha_3(\mu)$. Therefore, we subsequently set α_2 and α_1 explicitly to zero in our study of κ_f and then treat their effect as a perturbation.

Because the κ_f are assumed to be functions of the α_i alone, we can write the scale-changing evolution of $\kappa_f(\mu)$ as

$$\mu\frac{d}{d\mu}\kappa_f(\mu) = \sum_{i=1,2,3}\beta_i\frac{\partial\kappa_f(\mu)}{\partial\alpha_i} . \tag{10}$$

The left-hand side of (10) has been perturbatively calculated in the standard Higgs-boson case. Ignoring quark mixing, one finds for the top quark[4]

$$\mu\frac{d}{d\mu}\kappa_t(\mu) = \frac{9}{4\pi}\kappa_t^2 - \kappa_t\left[\frac{4}{\pi}\alpha_3 + \frac{9}{8\pi}\alpha_2 + \frac{17}{40\pi}\alpha_1 - \frac{3}{4\pi}\kappa_b - \frac{3}{2\pi}\sum_{q=u,d,s,c}\kappa_q - \frac{1}{2\pi}\sum_{l=e,\mu,\tau}\kappa_l\right] , \tag{11}$$

and similar expressions for light fermions.[4,14]

We next insert (11) into (10) and solve for κ_t. Since all κ_f with $f \neq t$ are very small, we set them to zero. For the reasons given above, we also initially set the smaller couplings α_2 and α_1 to zero and subsequently include their effect. With those simplifications, one finds

$$-\frac{7}{2\pi}\alpha_3^2\frac{d\kappa_t}{d\alpha_3} = \frac{9}{4\pi}\kappa_t^2 - \frac{4}{\pi}\alpha_3\kappa_t , \tag{12}$$

which is the nonlinear differential equation studied by KSZ.[4] They gave the general family of solutions

$$\kappa_t = \frac{2\alpha_3^{8/7}}{C + 9\alpha_3^{1/7}} , \tag{13}$$

with C an arbitrary constant. If we require κ_t to be continuous for all α_3, then $C < 0$ solutions (which correspond to very large m_t) are eliminated. All the solutions in (13) automatically vanish as $\alpha_3 \rightarrow 0$, but two particular solutions

$C=\infty$ and $C=0$ are special. The $C=\infty$ case corresponds to the ultraviolet-stable trivial solution $m_t=0$. The trivial solution is presumably chosen in lowest order for all fermions except the top quark. The second important solution, $C=0$, corresponds to

$$\kappa_t = \tfrac{2}{9}\alpha_3\,, \tag{14}$$

the solution advocated by KSZ[4] and PR.[6] It is infrared stable in that all $C\neq\infty$ solutions approach it for large α_3. That solution is also the unique nontrivial power-series solution about $\alpha_3=0$ and therefore represents a perturbative expansion in α_3. KSZ[4] chose the $C=0$ solution because it preserves perturbative renormalizability while PR[6] chose it because it corresponds to an infrared fixed point.[15]

We single out $C=0$ as the top-quark mass prediction because that solution is infrared stable and represents the unique power-series expansion about $\alpha_3=0$. It corresponds to the case in which both $m_t(\mu)$ and $m_W(\mu)$ asymptotically go continuously to zero as $\alpha_3(\mu)\to 0$. For the value of $\alpha_3(m_W)$ in (8), that lowest-order solution corresponds to $m_t\simeq 95.2$ GeV. There are a number of refinements that can be made. Treating the electroweak corrections as a perturbation on (14) gives an approximate shift

$$\kappa_t \simeq \tfrac{2}{9}\alpha_3 - \tfrac{1}{12}\alpha_2 - \tfrac{17}{540}\alpha_1\,, \tag{15}$$

which decreases the mass to 88.1 GeV. That is compensated by upward shifts of κ_t by about $(0.5/\pi)\alpha_3^2$ from two-loop contributions to (12) and $(16/27\pi)\alpha_3^2$ in going from the $\overline{\rm MS}$ definition of m_t to the physical mass. Together, they shift m_t back up to 96.3 GeV. Finally, we should actually use $\alpha_i(m_t)$ rather than $\alpha_i(m_W)$ in our calculation. That adjustment leads to the prediction[16]

$$m_t \simeq 95 \text{ GeV (with Higgs boson)}\,, \tag{16}$$

with an estimated error of about ± 5 GeV from the uncertainty in $\alpha_3(m_W)$. There is also uncertainty from our reliance on perturbative theory and ambiguities in electroweak effects.[7] KSZ[4] carried out a similar analysis for the Higgs-boson self-coupling. Updating that study gives $m_{\rm Higgs}\simeq 73$ GeV; but the uncertainty is very large because the electroweak shift is so big. (KSZ[4] found a smaller shift.)

The above prediction for the top-quark mass is interesting; but it has potential shortcomings. For example, embedding the standard model in a grand unified theory and applying the resulting renormalization-group operations along with our asymptotic conditions at unification can lead to a different m_t prediction. Also, the Higgs scheme is renormalizable and well defined (modulo triviality[17]) without additional constraints; so, the motivation is not so clear.

In our opinion, the above prescription is better suited for dynamical symmetry-breaking and mass-generation scenarios, i.e., no explicit Higgs boson, where the relationship between m_t and m_W can be viewed as a kind of self-consistency condition and it is natural to assume that κ_f varies continuously with the α_i. Even the results resemble earlier findings[9-11] in dynamical models.

To derive an analog of (11) for the case of dynamical symmetry breaking, we can self-consistently compute the one-loop corrections to α_2, m_t^2, and m_W^2 keeping fermion and gauge-boson masses in those calculations. That amounts to taking the standard-model calculation for $\mu\, d\kappa_t/d\mu$ and throwing away corrections due to the physical Higgs scalar. In that way we find the analog of (11) with $(9/4\pi)\kappa_t^2\to(3/2\pi)\kappa_t^2$ and $-(9/8\pi)\alpha_2\kappa_t\to-(3/4\pi)\alpha_2\kappa_t$. Again setting $\kappa_f=0$ for $f\neq t$ and $\alpha_2=\alpha_1=0$ leads to

$$-\frac{7}{2\pi}\alpha_3^2\frac{d\kappa_t}{d\alpha_3}=\frac{3}{2\pi}\kappa_t^2-\frac{4}{\pi}\alpha_3\kappa_t\,, \tag{17}$$

which can be analyzed in exactly the same manner as the Higgs-boson case. The counterpart of the $C=0$ solution in (14) is

$$\kappa_t = \tfrac{1}{3}\alpha_3 \text{ (dynamical)}\,. \tag{18}$$

We single it out as the nontrivial self-consistent solution because it is an infrared-stable perturbative expansion about $\alpha_3=0$. That solution corresponds to the case in which both $m_t(\mu)$ and $m_W(\mu)$ asymptotically go to zero.[18] Of course, to be physically stable, the solution in (18) should minimize the vacuum energy. We do not address that important issue or why $v\simeq 250$ GeV instead of say 100 MeV (the scale of ordinary chiral-symmetry breaking) is the electroweak scale. Since it seems unlikely that $\alpha_3(m_t)$ alone is strong enough to cause symmetry breaking at 250 GeV in the top sector, we must conjecture that some unknown new physics gives rise to dynamical mass generation for the top quark and also leads to gauge-boson masses.[9-11] It is interesting to note that at least three generations are required to generate a nontrivial, physically relevant, perturbative solution.

The lowest-order prediction in (18) corresponds to $m_t=116.6$ GeV. Including electroweak and QCD corrections, which again tend to cancel, gives

$$m_t \simeq 115 \text{ GeV (dynamical)}\,. \tag{19}$$

That prediction is about 20 GeV larger than the Higgs-boson case. If a very tightly bound pointlike 0^{++} $t\bar t$ state mimics the fundamental Higgs boson, we expect the dynamical prediction to be lowered somewhat towards the Higgs-boson prediction.

The result in (18) can also be derived using the weak-isospin-breaking top-quark self-energy (including QCD corrections)

$$\Sigma_t(p)\simeq m_t\left(\frac{\alpha_3(p)}{\alpha_3(m_t)}\right)^{4/7} \tag{20}$$

in the W-boson two-point function. Following the analysis in Refs. 9–11, one finds

$$m_W^2 \simeq -\frac{3i\alpha_2(m_W)}{8\pi^3}\int d^4p \frac{\Sigma_t^2}{p^2(p^2-\Sigma_t^2)} . \qquad (21)$$

Integrating that expression leads to the results in (4) and (18). This approach illustrates a sensitivity to new physics, even at very short distances. If we cut the integration off at some very large scale Λ, one finds from (21)

$$m_t^2 \simeq \frac{2}{3}\frac{m_W^2}{\alpha_2(m_W)}\frac{\alpha_3^{8/7}(m_t)}{\alpha_3^{1/7}(m_t)-\alpha_3^{1/7}(\Lambda)} . \qquad (22)$$

For $\Lambda \to \infty$, we get (18); however, even for $\Lambda \simeq 10^{15}$ GeV, where $\alpha_3(\Lambda) \simeq 0.023$, that expression gives $m_t \simeq 250$ GeV. So, new physics can have a significant influence on m_t and lead to a very heavy top quark. Another possibility is to have a fourth generation. In that event, m_t may be lowered somewhat due to the presence of heavier quarks in loop dynamics. Correspondingly, large t-t' mixing may be likely.[19]

In both the Higgs-boson and dynamical scenarios, a heavy top-quark mass is predicted. That has interesting experimental consequences. It suggests that finding the top quark at Fermilab's 1.8-TeV $p\bar{p}$ collider will take some time, but will not be difficult once somewhat higher luminosities are achieved. The production $p\bar{p} \to t\bar{t}+X$ via gluon-gluon scattering will lead to W^+W^- pairs in the final state from $t \to W+b$. Background from direct W^+W^- production is small; so, the heavy-top-quark signature will be very distinct.

If the top quark is responsible for or closely connected with $W^\pm$ and Z mass generation, it plays a special role. We will, therefore, want to explore its properties very thoroughly. It is likely to be the key to understanding all fermion masses and mixing.

Self-consistency may relate top-quark and W masses; but it does not explain their origin or the electroweak scale. It is hoped that ongoing and future experiments will give us the answers.

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Note added.—After the submission of this Letter, two new relevant papers were published. J. Kubo, K. Sibold, and W. Zimmermann [Phys. Lett. B **220**, 185 (1989)] have updated their earlier coupling-reduction results. V. Miransky, M. Tanabashi, and K. Yamawaki, [Phys. Lett. B **221**, 177 (1989)] have examined consequences of dynamical symmetry breaking by a large $t\bar{t}$ condensate.

[1] UA1 Collaboration, C. Albajar *et al.*, Z. Phys. C **37**, 505 (1988).

[2] ARGUS Collaboration, H. Albrecht *et al.*, Phys. Lett. B **192**, 245 (1987).

[3] U. Amaldi *et al.*, Phys. Rev. D **36**, 1385 (1987).

[4] J. Kubo, K. Sibold, and W. Zimmermann, Nucl. Phys. **B259**, 331 (1985).

[5] W. Zimmermann, Commun. Math. Phys. **97**, 211 (1985); R. Oehme and W. Zimmerman, *ibid.* **97**, 569 (1985).

[6] B. Pendleton and G. G. Ross, Phys. Lett. **98B**, 291 (1981).

[7] Electroweak corrections were found in Ref. 6 to shift the effective fixed point to higher m_t values. That is opposite in sign from the shift found in our analysis and the prescription of Ref. 4. Without a complete theory, electroweak effects are somewhat ambiguous.

[8] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964).

[9] R. Jackiw and K. Johnson, Phys. Rev. D **8**, 2386 (1973).

[10] J. Cornwall and R. Norton, Phys. Rev. D **8**, 3338 (1973); J. Cornwall, Phys. Rev. D **10**, 500 (1974).

[11] A. Carter and H. Pagels, Phys. Rev. Lett. **43**, 1845 (1979). See also F. Englert and R. Brout, Phys. Lett. **49B**, 77 (1974).

[12] We are using $\Lambda_{\overline{\rm MS}}^{(4)} = 160^{+120}_{-60}$ MeV and assuming $m_t \gtrsim m_W$ which is consistent with the final results.

[13] W. Marciano and A. Sirlin, in *Proceedings of the Second Workshop on Grand Unification,* edited by J. Leveille, L. Sulak, and D. Unger (Birkhauser, Boston, 1981), p. 151.

[14] M. Machacek and M. Vaughn, Nucl. Phys. **B236**, 221 (1984); K. Babu and E. Ma, Z. Phys. C **31**, 451 (1986).

[15] There is also a perturbative zero in Eq. (12) at $\kappa_t = \frac{16}{9}\alpha_3$ (i.e., $m_t \simeq 250$ GeV) where κ_t can get stuck while trying to reach its fixed point; see C. Hill, Phys. Rev. D **24**, 619 (1981).

[16] If the Higgs-scalar mass is larger than m_t, the prediction in Eq. (16) increases by $0.3\ln(m_H/m_t)$ GeV.

[17] D. Callaway, Phys. Rep. (to be published).

[18] Our statement that both $m_t(\mu)$ and $m_W(\mu)$ asymptotically go to zero takes into account only the behavior of those masses due to $SU(3)_C$. The top mass evolves as $m_t^2(\mu)/m_t^2 \sim \alpha_3(\mu)^{8/7}$ while $m_W^2(\mu)/m_W^2 \sim \alpha_3(\mu)^{1/7}$.

[19] If m_t is lighter than predicted, a heavier fourth generation could help dynamically generate W and Z masses. That case will be considered in a separate publication.