Gauge Invariance in Chern-Simons Theory on a Torus

Yutaka Hosotani[(a)]
Institute for Advanced Study, Princeton, New Jersey 08540
(Received 3 February 1989)

In Chern-Simons gauge theory on a manifold $T^2 \times R^1$ (two-torus×time) the unitary operators, which induced large gauge transformations shifting the nonintegrable phases of the two distinct Wilson-line integrals on the torus by multiples of 2π, do not commute with each other unless the coefficient of the Chern-Simons term is quantized. In U(1) theory this condition gives the statistics phase $\theta=\pi/n$ (n is an integer). The condition coincides with the one previously derived on a manifold S^3 (three-sphere) for SU($N\geq 3$) theory but differs by a factor of 2 for SU(2) theory. The requirement of the Z_N invariance in pure SU(N) gauge theory imposes a stronger constraint.

PACS numbers: 11.15.−q, 05.30.−d, 74.65.+n

In 2+1 dimensions one can always add to the Lagrangian the Chern-Simons term

$$\mathcal{L}_{\rm CS}^1=\tfrac{1}{2}\mu\epsilon^{\lambda\nu\rho}A_\lambda\partial_\nu A_\rho \tag{1}$$

in U(1) gauge theory, or

$$\mathcal{L}_{\rm CS}^2=\mu\epsilon^{\lambda\nu\rho}{\rm Tr}A_\lambda(\partial_\nu A_\rho+\tfrac{2}{3}igA_\nu A_\rho) \tag{2}$$

in non-Abelian gauge theory, where $A_\mu=A_\mu^aT^a$ and $[T^a,T^b]=if^{abc}T^c$ with the trace in the fundamental representation ${\rm Tr}T^aT^b=\frac{1}{2}\delta^{ab}$. It was previously introduced to generate a topological mass of gauge bosons.[1-4] More recently, it has been argued that the addition of (1) in U(1) theory leads to fractional statistics,[5] and could be essential to construct an effective theory for high-T_c superconductivity.[6] Also it has been shown that pure non-Abelian Chern-Simons theory is a powerful tool in exploring knot theory in mathematics,[7] and provides a new way of formulating theory of gravity in 2+1 dimensions.[8]

It is known that on a manifold S^3 (a three-sphere) the coefficient μ in (2) in non-Abelian gauge theory must be quantized in the unit of $g^2/4\pi$ so that the action may change only by multiples of 2π under large gauge transformations.[2] We consider a theory on a manifold $T^2\times R^1$ (two-torus×time) and derive a quantization condition for μ in both Abelian and non-Abelian theories. In addition to academic curiosity about properties of gauge theory on multiply connected space, putting a gauge theory on a torus has the advantage of eliminating the infrared ambiguity which quite often plagues analysis of gauge theory in Minkowski spacetime.

In gauge theory on a multiply connected space nonintegrable phases of the Wilson-line integrals along noncontractable loops become physical degrees of freedom.[9,10] Dynamics of such phases lead to rich physical consequences,[9-11] which, in general, do not disappear even in the infinite-volume limit. As an example, in QED on $S^1\times R^1$ (circle×time) the nonintegrable phase couples through the anomaly to the zero mode of fermion-antifermion bound states, leading to the θ vacuum.[11] In other words the structure of the θ vacuum is a direct consequence of the invariance of the theory under large gauge transformations. It is our hope that the analysis of Chern-Simons gauge theory on a torus, in its infinite-volume limit, gives crucial information on fractional statistics and high-T_c superconductivity.

We start to analyze a U(1) theory with the Lagrangian

$$\mathcal{L}_{\rm tot}=-\tfrac{1}{4}\kappa F_{\mu\nu}F^{\mu\nu}+\mathcal{L}_{\rm CS}^1+\mathcal{L}_{\rm mat}[A_\mu,\psi]\,, \tag{3}$$

on a torus ($0\leq x_j\leq L_j$, $j=1,2$). Since the space is multiply connected, one has to specify boundary conditions for the fields A_μ and ψ. After translations along noncontractible loops the fields need to return to their original values up to gauge transformations:

$$A_\mu[h_j(x)]=A_\mu[x]+\frac{1}{e}\partial_\mu\beta_j(x)\,,$$
$$\psi[h_j(x)]=e^{i\beta_j(x)}\psi[x]\,, \tag{4}$$

where $h_1(x)=(t,x_1+L_1,x_2)$ and $h_2(x)=(t,x_1,x_2+L_2)$. The most general β_j which is t independent and linear in x is given, up to gauge transformations, by

$$\beta_j(x)=-\epsilon^{jk}\pi a x_k/L_k\,, \tag{5}$$

where $\epsilon^{jk}=-\epsilon^{kj}$ ($\epsilon^{12}=1$). To guarantee $\psi[h_2(h_1(x))]=\psi[h_1(h_2(x))]$, the constant a must be an integer. It leads to the flux-quantization condition[12] $\Phi=\int d\mathbf{x}F_{12}=-2\pi a/e$.

The integer a is related, through one of the equations of motion,

$$\kappa\partial_\nu F^{\mu\nu}-\tfrac{1}{2}\mu\epsilon^{\mu\nu\lambda}F_{\nu\lambda}=eJ^\mu\,, \tag{6}$$

to the total charge

$$Q=\int d\mathbf{x}J^0=-\frac{\mu}{e}\Phi=\frac{2\pi\mu}{e^2}a\,. \tag{7}$$

As we shall see below, $2\pi\mu/e^2$ must be an integer ($\equiv n$) so that $Q=q$ must be a multiple of n ($q=an$). Gauge transformations, which respect (4), are

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \Lambda, \quad \psi' = e^{i\Lambda}\psi, \quad \Lambda = 2\pi\left(\frac{m_1 x_1}{L_1} + \frac{m_2 x_2}{L_2}\right) + \tilde{\Lambda}(t,\mathbf{x}). \tag{8}$$

Here m_1 and m_2 are integers, and $\tilde{\Lambda}(t,\mathbf{x})$ is a periodic function of $\mathbf{x}$.

First we consider the case $\kappa=0$, in which there exists no photon degree of freedom.[4] In the div$\mathbf{A}=0$ gauge,

$$\begin{aligned} A_0 &= \frac{e}{\mu}\int d\mathbf{y}\, D(\mathbf{x}-\mathbf{y})(\partial_1 J^2 - \partial_2 J^1)(t,\mathbf{y}), \\ eL_j A_j &= \theta_j(t) + \epsilon^{jk}\frac{e^2 q x_k}{2\mu L_k} + \frac{e^2 L_j}{\mu}\int d\mathbf{y}\, D(\mathbf{x}-\mathbf{y})\epsilon^{jk}\partial_k\left(J^0(t,\mathbf{y}) - \frac{q}{L_1 L_2}\right), \end{aligned} \tag{9}$$

where $\nabla^2 D(\mathbf{x}) = \delta(\mathbf{x})$ and $\int d\mathbf{x}\, D(\mathbf{x}) = 0$. θ_j's, the nonintegrable phases of the Wilson-line integrals $\exp(ie \times \int_0^{L_j} dx_j A_j)$, are the only physical gauge-field degrees of freedom. The residual gauge invariance in the $Q=0$ sector, for instance, is given by

$$\theta_j(t) \to \theta_j(t) + 2\pi m_j, \quad \psi_{n_1,n_2}(t) \to \psi_{n_1-m_1,n_2-m_2}(t), \tag{10}$$

where m_1 and m_2 are integers, and $\psi_{n_1,n_2}(t)$'s are Fourier components of $\psi(t,\mathbf{x})$.

Substitution of (9) into (3) yields the Lagrangian $=\mu\theta_2\dot{\theta}_1/e^2 + \cdots$ so that $\mu\theta_2/e^2$ is canonically conjugate to θ_1: $[\theta_1,\theta_2] = ie^2/\mu$. Therefore, the unitary operators, which generate the residual gauge transformations $(m_1,m_2) = (1,0)$ and $(0,1)$, are

$$U_j = \exp\left(+\epsilon^{jk}\frac{2\pi i\mu}{e^2}\theta_k\right)U_j^{\text{mat}}. \tag{11}$$

Here U_j^{mat}'s induce the shift in the matter fields. U_1 and U_2 commute with the Hamiltonian. However, since $U_1 U_2 = \exp(-4\pi^2 i\mu/e^2)U_2 U_1$, they commute with each other and states can be gauge invariant only if

$$\mu = \frac{e^2}{2\pi}n \quad (n \text{ is an integer}). \tag{12}$$

It is known[13] that in the presence of the Chern-Simons term the interchange (π rotation) of two identical particles gives Schrödinger wave functions an extra phase factor $e^{i\theta}$, where $\theta = e^2/2\mu$. Therefore, $\theta_{\text{stat}} = \pi/n$. A similar quantization condition has been previously derived[14] from the requirement of the gauge invariance in the presence of magnetic monopoles in R^3. Also it has been recently shown[15] that the modular invariance in (θ_1,θ_2) space is achieved only for an even integer n in (12).

The presence of the F^2 term in (3) does not affect the result. The relevant part of the Lagrangian is

$$\frac{\kappa}{2e^2}\left(\frac{L_2}{L_1}\dot{\theta}_1^2 + \frac{L_1}{L_2}\dot{\theta}_2^2\right) + \frac{\mu}{2e^2}(\theta_2\dot{\theta}_1 - \theta_1\dot{\theta}_2) + \cdots. \tag{13}$$

Conjugate momenta to θ_j's are

$$p_j = \frac{\kappa L_1 L_2}{e^2 L_j^2}\dot{\theta}_j + \epsilon^{jk}\frac{\mu}{2e^2}\theta_k. \tag{14}$$

They satisfy $[\theta_j,p_k] = i\delta_{jk}$. All other commutators vanish. This time

$$U_j = \exp\left[2\pi i\left(p_j + \epsilon^{jk}\frac{\mu}{2e^2}\theta_k\right)\right]U_j^{\text{mat}}. \tag{15}$$

The commutativity of U_1 and U_2 leads to the same quantization condition (12). In view of (14), (15) reduces to (11) in the $\kappa=0$ limit.

In SU(N) gauge theory we focus on a particular boundary condition $A_\mu[h_j(x)] = A_\mu[x]$. More general boundary conditions have been analyzed in Ref. 10. Then our boundary condition is invariant under gauge transformations $A_\mu \to \Omega A_\mu \Omega^\dagger - (i/g)\Omega\partial_\mu\Omega^\dagger$, provided that $\Omega[h_j(x)] = \Omega[x]$, or, in pure gauge-field theory, $\Omega[h_j(x)]\Omega[x]^\dagger$ is an element of the center of SU(N).

Let us consider pure SU(N) Chern-Simons theory: $\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{CS}}^2$. One of the equations gives a constraint $F_{12}=0$. Given an arbitrary single-valued A_1 in this subspace, the gauge transformation,

$$\begin{aligned} \Omega(x)^\dagger &= W(x)\exp[igx_1 B(t,x_2)], \\ W(x) &= P\exp\left(-ig\int_0^{x_1} dy_1 A_1(t,y_1,x_2)\right), \\ \exp[-igL_1 B(t,x_2)] &= W(t,L_1,x_2), \end{aligned}$$

which satisfies $\Omega[h_j(x)] = \Omega[x]$, brings $A_1(x)$ to $B(t,x_2)$, which in turn is diagonalized by a second x_1-independent gauge transformation. Then the constraint $F_{12}=0$ implies that A_1 is x_2 independent and A_2 also is diagonal and x_1 independent. A third gauge transformation with diagonal $\Omega = \Omega(t,x_2)$ can eliminate the x_2 dependence of A_2. Therefore, one can take without loss of generality,

$$gL_j A_j = \begin{pmatrix} \theta_{j1}(t) & & \\ & \ddots & \\ & & \theta_{jN}(t) \end{pmatrix}, \tag{16}$$

where $\sum_{a=1}^N \theta_{ja}(t) = 0$. A_0 is a dependent variable. Indeed, parts of the equations $F_{0j}=0$ with (16) imply that A_0 also is diagonal and depends only on t. A fourth gauge transformation with diagonal $\Omega = \Omega(t)$ then can gauge away A_0 entirely ($A_0 = 0$).

There are two kinds of residual gauge invariances.

One is

$$\Omega_{ab}=\delta_{ab}\exp\left[2\pi i\left(\frac{m_{1a}x_1}{L_1}+\frac{m_{2a}x_2}{L_2}\right)\right],\quad \theta_{ja}\rightarrow\theta_{ja}+2\pi m_{ja}\,, \tag{17}$$

where m_{ja}'s are integers satisfying $\sum_{a=1}^{N}m_{ja}=0$. The other is the Z_N transformation for which $m_{ja}=(1-N\delta_{ab})l_j/N$ $[a,b=1\text{-}N,\ l_j=1\text{-}(N-1)]$:

$$\theta_{ja}\rightarrow\theta_{ja}+2\pi l_j\left(\frac{1}{N}-\delta_{ab}\right). \tag{18}$$

This is a special symmetry in pure gauge-field theory.

Substitution of (16) and $A_0=0$ into $\mathcal{L}^2_{\rm CS}$ yields, in terms of θ_{ja} $[a=1\text{-}(N-1)]$,

$$L=\frac{2\mu}{g^2}\left(\sum_a{}'\theta_{2a}\dot\theta_{1a}+\sum_a{}'\theta_{2a}\sum_b{}'\dot\theta_{1b}\right),$$

where $\sum_a'=\sum_{a=1}^{N-1}$. Therefore, $p_{ja}=\epsilon^{jk}(2\mu/g^2)(\theta_{ka}+\sum_b'\theta_{kb})$ satisfies

$$[\theta_{ja},p_{kb}]=i\delta_{jk}\delta_{ab},\quad [\theta_{1a},\theta_{2b}]=i\frac{g^2}{2\mu}\left(\delta_{ab}-\frac{1}{N}\right),\quad [p_{1a},p_{2b}]=i\frac{2\mu}{g^2}(\delta_{ab}+1)\,, \tag{19}$$

with all other commutators vanishing.

The unitary operators $U_{ja}=\exp(2\pi i p_{ja})$ $[a=1\text{-}(N-1)]$, which generate (17), satisfy

$$U_{1a}U_{2b}=\exp\left(-\frac{8\pi^2 i\mu}{g^2}(\delta_{ab}+1)\right)U_{2b}U_{1a}\,, \tag{20}$$

so that the commutativity of U_{ja}'s leads to

$$\mu=\begin{cases}(g^2/8\pi)n, & \text{for SU(2)}\,,\\ (g^2/4\pi)n, & \text{for SU}(N\geq 3)\,,\end{cases} \tag{21}$$

where n is an integer. The condition (21) is the same as the one derived on a manifold S^3 in Ref. 2 for SU($N\geq 3$), but is weaker than that by a factor of 2 for SU(2). It is to be seen how the additional factor of 2 constraint arises in SU(2) theory on a torus.[15,16]

Equation (18) is generated by combinations of $\bar U_j=\exp[(2\pi i/N)\sum_a' p_{ja}]$ and U_{ja}. The requirement of the commutativity of these unitary operators leads to a stronger constraint:

$$\mu=\frac{Ng^2}{4\pi}n\quad (n\text{ is an integer})\,. \tag{22}$$

In other words, if μ satisfies (21) but not (22), then the Z_N symmetry is spontaneously broken.

In the presence of the F^2 term one cannot simultaneously diagonalize A_1 and A_2 in general. If one freezes all gauge-field degrees of freedom but the nonintegrable phases of the Wilson-line integrals, then one finds that conjugate momenta to θ_{ja} $(a=1,\ldots,N-1)$ are

$$p_{ja}=\frac{L_1L_2}{L_j^2}\frac{2\kappa}{g^2}\left(\dot\theta_{ja}+\sum_b{}'\dot\theta_{jb}\right)+\epsilon^{jk}\frac{\mu}{g^2}\left(\theta_{ka}+\sum_b{}'\theta_{kb}\right).$$

They satisfy $[\theta_{ja},p_{kb}]=i\delta_{jk}\delta_{ab}$. All other commutators vanish. The unitary operators generating (17) are

$$U_{ja}=\exp\left\{2\pi i\left[p_{ja}+\epsilon^{jk}\frac{\mu}{g^2}\left(\theta_{ka}+\sum_b{}'\theta_{kb}\right)\right]\right\}.$$

The commutativity of these operators leads to the same results as (20) and (21).

When μ obeys the quantization condition (12) or (21), it is meaningful to consider states which are gauge invariant up to a phase. In U(1) theory,

$$U_j\Psi_{\alpha_1\alpha_2}=e^{i\alpha_j}\Psi_{\alpha_1\alpha_2}\,. \tag{23}$$

In the $Q=0$ sector of nonrelativistic theory $\Psi=\Psi_{\rm gauge}\otimes|0\rangle_F$. For $\kappa=0$, θ_1 and θ_2 are canonically conjugate to each other, and the wave function of the state $\Psi_{\rm gauge}$ is

$$u(\theta_1)=\langle\theta_1|\Psi_{\rm gauge}\rangle\,,$$

$$v(\theta_2)=\langle\theta_2|\Psi_{\rm gauge}\rangle=\frac{\sqrt{n}}{2\pi}\int_{-\infty}^{+\infty}d\theta_1\,e^{-in\theta_1\theta_2/2\pi}u(\theta_1)\,.$$

For $\mu=e^2n/2\pi$, wave functions satisfying (23) are given by

$$u_{\alpha_1\alpha_2}(\theta_1)=e^{i\alpha_1\theta_1/2\pi}\delta_{2\pi}\left(\theta_1+\frac{1}{n}(\alpha_2+2\pi l)\right), \tag{24}$$

where $l=0,1,\ldots,n-1$ and $\delta_{2\pi}(\theta)$ is a periodic δ function with a period 2π. It is easy to check that $v(\theta_2)$ takes the same form as $u(\theta_1)$.

In the $Q=1$ sector of nonrelativistic theory (with neutralizing uniform background charge $Q_{\rm bg}=-1$) states satisfying (23) are

$$|\Psi\rangle=\int_{-\infty}^{+\infty}d\theta_1\sum_{m_1,m_2}e^{i\alpha_1\theta_1/2\pi+im_2(n\theta_1+\alpha_2)}h(\theta_1-2\pi m_1)\,|\theta_1\rangle\otimes|m_1,m_2\rangle\,,$$

where $|m_1,m_2\rangle=\psi^\dagger_{m_1m_2}|0\rangle_F$. $h(\theta)$ is an arbitrary function, and should be determined so as to solve the Schrödinger equation.

In SU(2) theory with $\mu = g^2 n/8\pi$ the structure of the commutation relations and the unitary operators are the same as in U(1) theory so that wave functions are given by (24) with the substitution $\theta_1 \rightarrow \theta_{11}$, $\alpha_j \rightarrow \alpha_{j1}$. In SU(3) theory with $\mu = g^2 n/4\pi$, states satisfying $U_{ja} | \Psi \rangle = e^{i\alpha_{ja}} | \Psi \rangle$ $(a=1,2)$ are

$$u(\theta_{11},\theta_{12}) = e^{i(\alpha_{11}\theta_{11}+\alpha_{12}\theta_{12})/2\pi} \delta_{2\pi}\left[\theta_{11} + \frac{1}{3n}(2\alpha_{21} - \alpha_{22} - 2\pi r) - \frac{2\pi q}{n}\right] \delta_{2\pi}\left[\theta_{12} + \frac{1}{3n}(-\alpha_{21} + 2\alpha_{22} - 2\pi r)\right],$$

where $r = 0,1,\ldots,3n-1$ and $q = 0,1,\ldots,n-1$. There are $3n^2$ states with given α_{ja}'s.

In this paper we have explored implications of the gauge invariance in Chern-Simons theory on a torus. The requirement of the gauge invariance has led to the quantization condition for the coefficient of the Chern-Simons term.

It is an interesting fact that the quantization condition follows even in U(1) theory, the case probably most important in physical applications. It is, however, a dynamical question whether or not the gauge invariance remains unbroken in the ground state. Moreover, one might wonder how the quantization condition derived on a torus has any relevance in physics in the Minkowski spacetime. It is quite likely that something very special happens in the Minkowski spacetime when the quantization condition is satisfied.[17] The experience in the analysis of QED on a circle[11] also suggests that as a consequence of the gauge invariance the wave function of the ground state with matter has the θ-vacuum structure, which should remain intact in the infinite-volume limit. If this is the case, the notion of the gauge invariance has to play an important role in discussing fractional statistics and high-T_c superconductivity.

This research was supported in part by DOE Contract No. DE-AC02-83ER-40105 and by the McKnight-Land grant at the University of Minnesota. The author would like to thank the Institute for Advanced Study for its hospitality where this work was done, and J. Hetrick, C-L. Ho, N. Seiberg, and F. Wilczek for stimulating discussions.

[a]On leave of absence from School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455.

[1]J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981).

[2]S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1983); Ann. Phys. (N.Y.) **140**, 372 (1984).

[3]A. N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); Phys. Rev. D **29**, 2366 (1984); R. Jackiw, Phys. Rev. D **29**, 2375 (1984).

[4]C. R. Hagen, Ann. Phys. (N.Y.) **157**, 342 (1984).

[5]F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982); F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983); D. P. Arovas, R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. **B251**, 117 (1985); A. Goldhaber, R. MacKenzie, and F. Wilczek, Harvard University Report No. HUTP-88/A044 (to be published); X. G. Wen and A. Zee, Santa Barbara Report No. NSF-ITP-88-114 (to be published); J. Fröhlich and P. A. Marchetti, "Quantum field theories of vortices and anyons," Eidgenössische Technische Hochschule University (to be published).

[6]P. W. Anderson, Science **235**, 1196 (1987); S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987); V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987); "Theory of the spin liquid state of the Heisenberg antiferromagnet," Stanford University (to be published); I. Dzyaloshinskii, A. Polyakov, and P. Wiegmann, Phys. Lett. A **127**, 112 (1988); A. M. Polyakov, Mod. Phys. Lett. A **3**, 325 (1988); J. March-Russel and F. Wilczek, Phys. Rev. Lett. **61**, 2066 (1988); R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988); X. G. Wen, F. Wilczek, and A. Zee, Santa Barbara Report No. NSF-ITP-88-179 (to be published).

[7]E. Witten, Institute for Advanced Study Report No. IASSNS-HEP-88/33 (to be published).

[8]E. Witten, Institute for Advanced Study Reports No. IASSNS-HEP-88/32, No. -88/55, and No. -89/1 (to be published).

[9]Y. Hosotani, Phys. Lett. B **126**, 309 (1983); D. Tom, Phys. Lett. B **126**, 445 (1983).

[10]Y. Hosotani, University of Minnesota Report No. UMN-TH-662/88 (to be published), and references therein.

[11]N. S. Manton, Ann. Phys. (N.Y.) **159**, 220 (1985); J. E. Hetrick and Y. Hosotani, Phys. Rev. D **38**, 2621 (1988).

[12]F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).

[13]A. S. Goldhaber *et al.*, in Ref. 5.

[14]O. Alvarez, Commun. Math. Phys. **100**, 279 (1985); M. Henneaux and C. Teitelboim, Phys. Rev. Lett. **56**, 689 (1986); R. D. Pisarski, Phys. Rev. D **34**, 3851 (1986).

[15]S. Elitzer, G. Moore, A. Schwimmer, and N. Seiberg, Institute for Advanced Study Report No. IASSNS-HEP-89/20 (to be published).

[16]G. Moore and N. Seiberg, Institute for Advanced Study Report No. IASSNS-HEP-89/6 (to be published); G. V. Dunne, R. Jackiw, and C. A. Trugenberger, Massachusetts Institute of Technology Report No. CTP-1711 (to be published).

[17]F. Wilczek (private communication).