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# Quantum Level Statistics of Pseudointegrable Billiards

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We study the spectral statistics of systems of two-dimensional pseudointegrable billiards. These systems are classically nonergodic, but nonseparable. It is found that such systems possess quantum spectra which are closely simulated by the Gaussian orthogonal ensemble. We discuss the implications of these results on the conjectured relation between classical chaos and quantum level statistics. We emphasize the importance of the semiclassical nature of any such relation.

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The statistical study of quantum level spectra has attracted much attention in recent years in conjunction with the speculation that it may reflect the degree of order in the corresponding classical system. Based on the study of simple low-dimensional systems,[1-8] it has been conjectured that the quantum level statistics of classically orderly systems show the characteristics of an uncorrelated random distribution (Poisson distribution) while the quantum spectra of classically chaotic systems are characterized by a Gaussian orthogonal ensemble (GOE). Using a model Hamiltonian which has both chaotic and nonchaotic orbits, Seligman, Verbaarschot, and Zirnbauer[5,6] showed that there is continuous transition between these two extremes, and suggested that properties of the level statistics are universal in the sense that some parameter which characterizes the degree of order in the classical system (e.g., the Kolmogorov entropy) also determines the statistical characteristics of the quantum spectral distribution.

It is well known that certain integrable systems do not have Poisson statistics,[1,9] defying the universal association of the two. It is also known that systems can be found which are classically chaotic but which have many different types of non-GOE quantum level statistics; the pseudospherical billiards of Refs. 10–12 represent samples of such systems. It is interesting to see whether there exist systems which violate the supposed connection between GOE statistics and classical chaos in the opposite manner, i.e., systems which have GOE level statistics but which correspond to classically nonchaotic physics. A particularly interesting billiard problem is the one studied by Richens and Berry.[13] The classical system has zero Kolmogorov entropy, but the quantum spectra show a non-Poisson distribution with level repulsion. The system has a property which is named pseudointegrability—the existence of two separately conserved energies despite the nonseparable nature of the system. This is possible because of the singular nature of billiard potentials, namely, the sharp edge and the infinite height of walls. The existence of the two conserved quantities implies that the system cannot be ergodic in phase space and the flat and rectangular nature of the walls implies a zero Liapunov exponent for all orbits.

In this Letter, we perform a numerical study of a set of generalized Richens-Berry billiards. Our model consists of a square billiard with a number of rectangular pieces, which we refer to as boxes, removed from a corner. This system is also pseudointegrable. We found that the spectral distribution is already close to that of the GOE even in the case of a single box inside the square well (the example of Ref. 13), and with increasing number of boxes one can make it arbitrarily close to the GOE. In a sense, it gives a counterexample to the proposed universality of the level statistics, at least when the Liapunov exponent and other measures of classical chaos are defined in a conventional way.

Billiard problems are useful both for the geometrical intuition and for the computational advantage of a sharp boundary which provides a natural choice for the basis wave function. Our model is motivated by an attempt to approximate the chaotic Sinai billiard[14] by a pseudointegrable one. In the model, a point particle with mass $m$ moves inside the square well of size $L$ of infinite height. On one corner of the well, there is an additional structure. The construction of the structure is shown in Fig. 1. We imagine a square of size $R$ $(<L)$ sharing two sides with the outer well, and the square is divided into $N^2$ equal cells (or boxes) of size $R/N$. We draw a circle of radius $R$ centered on the common corner of the inner square and the outer well, and place the potential wall along the boundary of cells in such a way that the wall traces the semicircle as closely as possible with a given number of the cell division $N$. With the two-dimensional coordinate shown in Fig. 1, the motion of the particle in the billiard is formally described by the Hamiltonian

$$H=\frac{1}{2m}[p_x^2+p_y^2]+V_0+\sum_{ij}V_{ij}\,, \tag{1}$$

where $p_x$ and $p_y$ denote the $x$- and $y$-axis components of the momentum. The potential energy of the outer well $V_0$ is given by

$$V_0=0\ \ (0\le x<L\ \text{and}\ 0\le y<L)\,,$$
$$V_0=\infty\ \ (\text{otherwise})\,, \tag{2}$$

and the potential of each cell $V_{ij}$ is given by

$$V_{ij}=\infty\ \ \left(\frac{i-1}{N}R\le x<\frac{i}{N}R\ \text{and}\ \frac{j-1}{N}R\le y<\frac{j}{N}R\right),$$
$$V_{ij}=0\ \ (\text{otherwise})\,. \tag{3}$$

The summation $i,j$ in Eq. (1) runs for the boxes outside the boundary. At the large-$N$ limit, our system approaches the Sinai billiard.[14] The $N=1$ case was already discussed by Richens and Berry[13] as an example of a nonergodic system which has level repulsion.

Clearly, the absolute value of two momentum projections of a particle to $x$ and $y$ axes are separately conserved, and the system is nonergodic. The Liapunov exponent $\Lambda$, which is defined as

$$\Lambda=\lim_{t\to\infty,\,d(0)\to 0}\frac{1}{t}\ln\left(\frac{d(t)}{d(0)}\right), \tag{4}$$

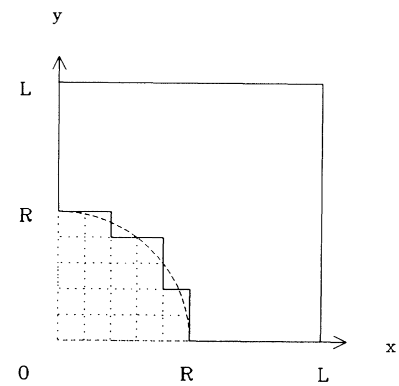


FIG. 1. The construction of the corner structure of the billiard. The division number in this case is $N=5$.

where $d$ is the distance between two orbits as the function of time $t$, is trivially zero for all orbits except those of a set of measure zero which hit a corner in a finite time. The Kolmogorov entropy $K$ is defined as the average value of the Liapunov exponent over all orbits at fixed energy. For our billiard, $K$ is automatically zero at all energy.

We calculate the spectra for the quantum system by straightforward diagonalization using the Fourier basis states for a square billiard of length $L$. In actual calculations, we have to replace the infinite potential in Eq. (3) by finite strength $s$. To correctly simulate the infinite height potential, the strength $s$ must be substantially larger than the maximum kinetic energy in a given truncated basis $T_{\max}$. On the other hand, $s$ cannot be too large in order not to cause round-off error. We found $s\sim 100T_{\max}$ to be a good compromise. Our results were found to be numerically stable against variations of $s$. After some numerical experiments, we found that for the billiard of the ratio $R/L=0.1$ to 0.5, truncating at about 2500 states is sufficient to obtain about 300 lowest eigenvalues with an accuracy better than 1%. The error in the difference between two adjacent eigenvalues was typically better than 5%. As usual the states are desymmetrized in order to remove the degeneracy caused by the reflective symmetry $(x\leftrightarrow y)$ of the system.[3] This corresponds to only considering states which are antisymmetric with respect to this symmetry in our statistics.

We focus on two statistical quantities, namely, the distribution of nearest-neighbor spacing $P_D$ and the Dyson-Mehta rigidity $\Delta_3$, which are defined[15] by

$$P_D(x)dx=\frac{\text{number of adjacent pair levels } (j,j+1) \text{ in which } x<(E_{j+1}-E_j)/D<x+dx}{\text{total number of adjacent pair levels sampled}}\,, \tag{5}$$

where $D$ is the average level spacing, and

$$\Delta_3(L)=\frac{1}{L}\left\langle \min_{A,B}\int_{E-L/2}^{E+L/2}dE'[n(E')-AE'-B]\right\rangle_E\,, \tag{6}$$

where $n(E)$ is the cumulative number of levels from ground state to energy $E$. The symbol $\langle\,\rangle_E$ stands for the averaging over energy $E$. Note that for two-dimensional billiards the average level spacing $D$ is constant as a function of $E$ except for the very low $E$ region.[16] In order to increase the statistics, we include systems with different size boxes. After having observed that $P_D$ and $\Delta_3$ are roughly invariant with respect to the size of the boxes for a few hundred low-lying states, we collected 200 lowest states (after discarding 50 lowest states to insure the constancy of average level spacing) from four different sizes of boxes $R=0.2L$, $0.3L$, $0.4L$, and $0.5L$. The results are shown in Fig. 2 for three examples $N=1$, 8, and 16. Solid lines in the figures are the predictions of the Gaussian orthogonal ensemble. One can see from the figures that even the single-box billiard shows remarkable similarity to the GOE distribution. The level statistics of the $N=16$ billiard are virtually indistinguishable from the GOE in the bin scale given in the figure and within the statistical fluctuations. The shape change of $P_D$ and $\Delta_3$ with the variation of $N$ is found to be smooth; for example, the values between $N=8$ and 16 give intermediate results both for $P_D$ and for $\Delta_3$.

Our principal result is that a quantum system whose classical analog is nonchaotic can have level statistics which are well approximated by the GOE. It is important to note that as we increase $N$ the spectrum is increasingly well approximated by GOE statistics. This is true despite the fact that the standard measures of classical chaos, such as the Kolmogorov entropy, do not increase. A sensible interpretation of this result calls for a careful analysis of the limiting process $N\rightarrow\infty$ and of the semiclassical limit. Clearly for any finite $N$, the classical system is not chaotic. However, it is not at all clear whether the classical mechanics is even well defined for $N\rightarrow\infty$. The quantum mechanics, on the other hand, is well defined, and is equivalent to the Sinai problem at this limit.

It has been argued that any relationship between quantum statistics and classical motion can only be valid in the semiclassical regime.[17] The semiclassical limit may be defined as the limit in which the typical variations of the quantum wave functions, i.e., $1/k$ where $\hbar k$ is the typical momentum, are much shorter than any of the relevant scales in the problem. It is plausible that in the semiclassical limit, the level statistics does reflect the degrees of order in the classical system. One demonstration of the need for the semiclassical limit is the behavior of the spectrum of the Sinai billiard. In the small-$kR$ limit, where a perturbative treatment is valid,[18] one can show that the low-lying spectra are characterized by a non-GOE spectrum close to the integrable spectrum for a blank billiard (i.e., square box). The semiclassical spectrum ($kR\gg 1$), however, is GOE.[3] For the pseudointegrable system being considered here, one must go to energies which give wavelengths that are very small compared to the size of the boxes at a corner in order to reach the semiclassical limit. Note that in the $N\rightarrow\infty$

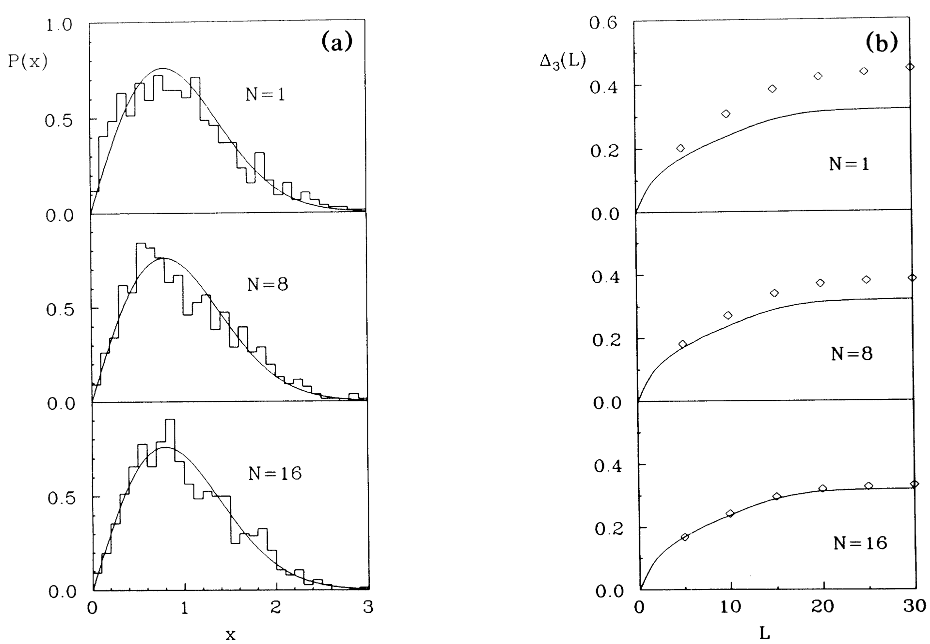


FIG. 2. (a) Nearest-level spacing distribution $P_D(x)$ for the level spectra of the square billiards $N=1$, 8, and 16. Solid lines are the predictions of the Gaussian orthogonal ensemble. (b) Dyson-Mehta statistics $\Delta_3(L)$ for the level spectra of the square billiard $N=1$, 8, and 16. Solid lines are the predictions of the Gaussian orthogonal ensemble.

limit the energy at which semiclassical physics sets in gets pushed off to infinity. The energy levels we considered in our system are not in the semiclassical regime since the wavelengths for the highest-energy states included in our statistics are larger than the box size. In this case one expects that the observables cannot depend on the sharpness of the edges of boxes since the edges cannot be resolved. Thus as far as the quantum system is concerned there is very little difference between our pseudointegrable system and the chaotic Sinai billiard problem.

If the preceding analysis is correct, the fact that our model spectra closely resemble GOE statistics does not contradict the proposed association of GOE statistics with chaos in the corresponding classical physics providing one restricts this association to semiclassical quantum spectra. We note, however, that our result does have practical significance because in nature all systems are quantal, and all the information one can obtain is ultimately about the various matrix elements. In most situations, one cannot tell whether one is truly in the semiclassical regime just by looking at the level spectra. Thus, if the quantum spectrum appears to be GOE one cannot automatically conclude that the corresponding classical system is chaotic.

Finally, we note that there might be a way to define a parameter which controls the characteristics of the quantum level statistics by incorporating the uncertainty principle in the Liapunov experiment. Normally, one judges the degree of chaos of an orbit by studying whether a pair of close orbits diverge exponentially. This is the meaning of the $d \to 0$ limit in the definition of $\Lambda$ in Eq. (4). It is reasonable, however, when trying to discern the effect of the classical mechanics on quantum physics, to only look at scales which can be resolved by quantum wave functions. Thus, one may want to consider how two orbits diverge which are spatially separated by distances comparable to the wavelength. Such orbits can diverge faster than any power law. In such a circumstance, within a finite time, one orbit can hit one step while its neighbor can hit a different step leading to a large separation over time. Thus one sees that at a coarse-grained level the system is essentially chaotic and the fact that it closely resembles a GOE spectrum may not be too surprising.

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[1]M. V. Berry and M. Tabor, Proc. Roy. Soc. London A **356**, 375 (1977).

[2]*Quantum Chaos and Statistical Nuclear Physics*, edited by T. H. Seligman and H. Nishioka (Springer-Verlag, Berlin, 1986).

[3]O. Bohigas, M. J. Giannoni, and C. Schmidt, Phys. Rev. Lett. **52**, 1 (1984).

[4]D. C. Meredith, S. E. Koonin, and M. R. Zirnbauer, Phys. Rev. A **37**, 3499 (1988).

[5]T. H. Seligman, J. J. M. Verbaarschot, and M. R. Zirnbauer, Phys. Rev. Lett. **53**, 215 (1985).

[6]T. H. Seligman, J. J. M. Verbaarschot, and M. R. Zirnbauer, J. Phys. A **18**, 2715 (1985).

[7]T. H. Seligman, J. J. M. Verbaarschot, and H. A. Weidenmüller, Phys. Lett. **167B**, 365 (1986).

[8]V. Parr and D. Vorkapic, Phys. Lett. B **205**, 7 (1988).

[9]G. Casati, B. V. Chrikov, and I. Guarneri, Phys. Rev. Lett. **54**, 1350 (1985).

[10]N. L. Balazs and A. Voros, Phys. Rep. **143**, 109 (1986).

[11]N. L. Balazs and A. Voros, Europhys. Lett. **4**, 1089 (1987).

[12]N. L. Balazs, C. Schmit, and A. Voros, J. Stat. Phys. **46**, 1067 (1987).

[13]P. J. Richens and M. V. Berry, Physica (Amsterdam) **2D**, 495 (1981).

[14]Ya. G. Sinai, Russ. Math. Surv. **25**, 137 (1970).

[15]M. L. Mehta, *Random Matrices* (Academia, New York, 1967).

[16]H. P. Baltes and E. R. Hilf, *Spectra of Finite System* (Wissenschaftsverlag, Manheim, 1976).

[17]M. V. Berry, in *The Wave-Particle Dualism*, edited by S. Deiner *et al.* (Reidel, Dordrecht, 1984), p. 231.

[18]M. V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981).