Analytic Expression for Mode Conversion of Langmuir and Electromagnetic Waves

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We have derived analytic expressions, in terms of Airy functions, for the reflection and mode-conversion coefficients of Langmuir and electromagnetic waves in an inhomogeneous, unmagnetized plasma. The reflection coefficient for the "direct" problem (incident electromagnetic wave) is equal in magnitude to that for the "inverse" problem (incident electrostatic wave) and the corresponding mode-conversion coefficients satisfy energy conservation. Our results, which are valid in the limit of nonrelativistic electron temperature, $T/mc^2 \ll 1$, agree with earlier numerical calculations.

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The coupling of electromagnetic and electrostatic waves in an inhomogeneous, unmagnetized plasma has been studied extensively.[1-10] An electromagnetic wave of frequency ω incident from the low-density region, with wave vector $\mathbf{k}$ inclined at an angle Θ with the density gradient, is partially reflected at the electromagnetic cutoff, where $\omega_p(z) = \omega\cos\Theta$, and partially mode converted to an electrostatic Langmuir wave at the plasma resonance point, where $\omega_p(0) = \omega$. This "direct" problem is closely related to the "inverse" problem of an obliquely incident Langmuir wave, which is partially reflected at its cutoff and partially converted to an electromagnetic wave. The direct problem is of interest because it describes an absorption mechanism for plasma heating by electromagnetic waves, e.g., in laser fusion. The inverse problem has bearing on the question of radiative loss of energy from electrostatic waves and on the remote detection of electrostatic wave activity, e.g., in simulated ionospheric emissions.[11]

The first analytic solution for a warm plasma was given by Piliya.[3] Baños and Kelly[5] used a combination of analytic and numerical techniques but obtained a result differing significantly from Piliya's. A numerical integration of the field equations for the direct problem by Forslund *et al.*[6] gave results in close agreement with Baños and Kelly[5] and this result is generally accepted as correct. In applications of these mode-conversion processes, it is often helpful to have analytic expressions for the absorption and reflection coefficients. Approximate analytic expressions were derived by Speziale and Catto[7] for the limiting cases of very small and very large values of $q \equiv (k_0L)^{2/3}\sin^2\Theta$, where $k_0 = \omega/c$ and L is the density gradient scale length, with c the speed of light. Pert[8] obtained power-series solutions valid for all q, but these, of course, require numerical evaluation for each specific case.

We present here analytic expressions for the mode conversion and reflection coefficients for both the direct and inverse problems in terms of Airy functions, valid to lowest order in an expansion in T/mc^2, where T is the electron temperature. Our results agree closely with the numerical solutions obtained previously[5,6,8] for all values of q. We show explicitly that the mode-conversion coefficients for the direct and inverse problems are equal, thus validating results obtained from time-reversal symmetry arguments.[9] The analytic expressions given here should be of use in the applications, and the technique used to derive them should be useful for more complicated mode-conversion problems. In particular, our analytic expressions for the reflection and mode-conversion coefficients provide a check for mode-conversion calculations in a magnetized plasma[12] in the limit of zero magnetic field.

We treat the electrons as a warm fluid whose unperturbed density near the reflection layer, $n_0 = \bar{n}(1 - z/L)$, coincides with that of the static ions. The density gradient is assumed to be the result of ionization, recombination, and loss processes as occurs, for example, in the ionosphere. Thus, unlike Forslund *et al.*[6] and Means *et al.*,[9] we do not include an ambipolar electric field $\mathbf{E}_0$ to balance the pressure gradient force. We linearize the fluid equations, setting $n(\mathbf{r},t) = n_0 + \{n_1(z)\exp[i(k_x x - \omega t)] + \text{c.c.}\}$, etc., and assume $pn^{-\gamma} = \text{const}$, with $\gamma = 3$, as is appropriate for high phase velocity waves. The continuity and momentum equations, together with Poisson's equation $\nabla\cdot\mathbf{E}_1 = 4\pi q n_1 \equiv k_0\rho$, give directly

$$\rho'' + K_s^2\rho = g'E_z/\beta^2\,, \tag{1}$$

where the prime denotes $d/d\bar{z}$ with $\bar{z} \equiv k_0 z$, $\beta^2 \equiv \gamma T/mc^2$, $K_s^2 \equiv (1-g)/\beta^2 - n_x^2$, $n_x \equiv k_x c/\omega$, $g(\bar{z}) \equiv \omega_p^2/\omega^2 = 1 - \bar{z}/k_0L$, and we have dropped the subscript 1 on E_1. The momentum equation gives for the current density,

$$\mathbf{J} = -n_0 e\mathbf{u} = (i\omega/4\pi)[g\mathbf{E} - \beta^2\nabla\rho/k_0]\,. \tag{2}$$

Substituting this into Ampere's law yields

$$(g-1)\mathbf{E} = \hat{\mathbf{x}}(B' + in_x\beta^2\rho) + \hat{\mathbf{z}}(\beta^2\rho' - in_xB)\,, \tag{3}$$

and taking the curl of (3) yields

$$B'' + K_m^2 B = g'E_x\,, \tag{4}$$

where $B_y \equiv -iB$ is the magnetic field and $K_m^2 \equiv 1-g-n_x^2$. Equations (1) and (4) show how the inhomogeneity couples the two amplitudes, ρ for the electrostatic wave and B for the electromagnetic wave. Closed equations for ρ and B can be obtained by using (3) to express the E_x and E_z, which appear on the right-hand sides of (1) and (4), in terms of ρ, B, and their derivatives, but we have been unable to find analytic solutions to the resulting equations. If, instead, we eliminate only E_x and retain E_z as an auxiliary variable, analytic solutions in terms of Airy functions can be obtained. Using (3) as well as $\rho = in_xE_x + E_z'$ and $B = E_x' - in_xE_z$ leads to a differential equation for E_z,

$$E_z'' + K_s^2 E_z = in_x(1-\beta^2)B/\beta^2 , \tag{5}$$

and an expression for E_x,

$$E_x = K_m^{-2}\{in_x[(1-\beta^2)\rho - E_z'] - B'\} , \tag{6}$$

which can be substituted into (4). We thus have three equations for ρ, B, and E_z, namely, (1), (4), and (5), with E_x given by (6). It is convenient to express them in terms of appropriately scaled coordinates and wave numbers: Z and N for the electromagnetic components, and ζ and n for the electrostatic components, where $Z \equiv (k_0L)^{-1/3}z = \beta^{2/3}\zeta$ and $N^2 = (k_0L)^{2/3}n_x^2 = q = \beta^{-4/3}n^2$. This yields

$$d^2E_z/d\zeta^2 + (\zeta - n^2)E_z = bB(z) , \tag{7}$$

$$d^2\rho/d\zeta^2 + (\zeta - n^2)\rho = -E_z/\alpha , \tag{8}$$

$$d^2B/dZ^2 - (Z-N^2)^{-1}dB/dZ + (Z-N^2)B = S , \tag{9}$$

where $b \equiv i\alpha N(1-\beta^2)\beta^{-4/3}$, $\alpha \equiv (k_0L\beta^2)^{1/3}$, $S = S(Z) = -iNQ(Z-N^2)^{-1}$ with $Q = (1-\beta^2)\rho - \alpha^{-1}dE_z/d\zeta$. To obtain an analytic solution of these equations, valid for small β, we exploit the great disparity (a factor of $\beta^{2/3}$) between the electromagnetic and electrostatic scales. Thus, we evaluate B on the right-hand side of (7) at $Z = \zeta = 0$, which reduces (7) to an inhomogeneous Airy equation for E_z.

We consider first the inverse problem. For this case, ρ and E_z must be bounded for $\zeta \to -\infty$ and may have both incident and outgoing components for $\zeta \to +\infty$. A convenient solution of (7) which satisfies these boundary conditions is

$$E_z(\zeta) = \overline{\mathrm{Ai}} + e_1[\overline{\mathrm{Gi}} + i\overline{\mathrm{Ai}}] , \tag{10}$$

where[13] $\overline{\mathrm{Ai}} = \overline{\mathrm{Ai}}(\zeta - n^2) \equiv \mathrm{Ai}(n^2 - \zeta)$ and $\overline{\mathrm{Gi}} = \overline{\mathrm{Gi}}(\zeta - n^2) \equiv \mathrm{Gi}(n^2-\zeta)$, and $e_1 = -\pi bB(0)$. (The amplitude of E_z of the incident Langmuir wave in this case would be $\frac{1}{2}$.) From (10) and the asymptotic forms of $\overline{\mathrm{Ai}}$ and $\overline{\mathrm{Gi}}$,[13] it follows that the ratio of reflected to incident amplitudes is $R = 1 + 2ie_1$. With E_z given by (10), we can solve (8) exactly as follows:

$$\rho(\zeta) = \rho_0\overline{\mathrm{Ai}} + \alpha^{-1}dE_z/d\zeta , \tag{11}$$

where ρ_0 is a constant to be determined.

To solve (9), we use the solutions of the homogeneous equation, namely, derivatives of Airy functions, to construct a Green's-function solution. For the inverse problem B must be bounded for $Z \to -\infty$ and have no incoming wave component for $Z \to +\infty$, so we can write

$$B(Z) = B_2(Z)\int_{-\infty}^{Z} dZ' B_1 S/W + B_1(Z)\int_{Z}^{\infty} dZ' B_2 S/W , \tag{12}$$

where $B_1(Z) \equiv \overline{\mathrm{Ai}}'(Z-N^2)$, $B_2(Z) \equiv \overline{A}'_+(Z-N^2)$, $\overline{A}_+(Z) \equiv \overline{\mathrm{Bi}}(Z) + i\overline{\mathrm{Ai}}(Z)$, and the Wronskian $W(Z') \equiv -(Z'-N^2)/\pi$. The two factors of $Z'-N^2$ in the denominator of the integrands in (12) can be removed by integration by parts using $B_1' = (N^2-Z')\overline{\mathrm{Ai}}$, $B_2' = (N^2 - Z')A_+$, and the relation

$$dQ/d\zeta = \beta^{4/3}(Z-N^2)E_z/\alpha , \tag{13}$$

which can be derived using (7) and the $\hat{\mathbf{z}}$ component of (3). We simplify the resulting expression for $B(Z)$ by again using the disparity of scale lengths, replacing, for example, $\int d\zeta' E_z(\zeta')\overline{\mathrm{Ai}}'(Z'-N^2)$ by $\overline{\mathrm{Ai}}'(-N^2)\int d\zeta' \times E_z(\zeta')$. Evaluating the resulting expression for $B(Z)$ at $Z=0$ yields a relation between $B(0)$ and ρ_0. A second relation between these quantities is obtained by evaluating the $\hat{\mathbf{z}}$ component of (3) at $Z=0$. It follows from these two relations that ρ_0 is of order $\beta^{4/3}$ and hence makes only a higher-order contribution to $B(Z)$. Then, to lowest order in β,

$$B(Z) = i\pi N\beta^{4/3}\alpha^{-1}\left[B_1(0)B_2(Z)\int_{-\infty}^{\zeta} d\zeta' + B_2(0)B_1(Z)\int_{\zeta}^{\infty} d\zeta'\right]E_z(\zeta') . \tag{14}$$

Evaluating (14) at $Z=0$ and making use of (10) gives

$$B(0) = i\pi N\beta^{4/3}\alpha^{-1}\Gamma(1 - i\pi^2 q\Gamma)^{-1} , \tag{15}$$

where $\Gamma = \mathrm{Ai}'(q)A'_+(q)$. From $R = 1 + 2ie_1 = 1 - 2i\pi b \times B(0)$, we have

$$R = (1 + i\pi^2 q\Gamma)(1 - i\pi^2 q\Gamma)^{-1} . \tag{16}$$

The mode-conversion coefficient, $|\eta|^2$, is the ratio of the outgoing electromagnetic energy flux $S_{\mathrm{em}}^{\mathrm{out}}$ and the incident electrostatic energy flux $S_{\mathrm{es}}^{\mathrm{in}}$. The total energy flux,

$$\mathbf{S} = 2\,\mathrm{Re}[(c/4\pi)\mathbf{E}_1 \times \mathbf{B}_1^* + p_1\mathbf{u}_1^*] , \tag{17}$$

satisfies $\nabla \cdot \mathbf{S} = 0$. Using (3), the $\hat{\mathbf{z}}$ component of the momentum fluid equation and $p_1 = \gamma n_1 T$ we have to lowest order in β,

$$S_z = c(k_0L)^{1/3}\,\mathrm{Im}(Z^{-1}B^*\,dB/dZ + \beta^{8/3}\zeta^{-1}\rho^*\,d\rho/d\zeta)/2\pi . \tag{18}$$

Using the large-z expressions for $B(Z)$ and $\rho(\zeta)$, we find

$$|\eta|^2 = q|2\pi \mathrm{Ai}'(q)|^2 |1 - i\pi^2 q\Gamma|^{-2}, \quad (19)$$

where $|\eta|^2 = S_{\mathrm{em}}^{\mathrm{out}}/S_{\mathrm{es}}^{\mathrm{in}}$. It is readily verified that energy is conserved, $|R|^2 + |\eta|^2 = 1$.

We now turn to the direct problem, which is somewhat easier to solve than the inverse problem. The basic equations (7), (8), and (9) remain the same but the boundary conditions are changed: For $Z \rightarrow \infty$, $B(Z)$ has both incident and reflected components, whereas ρ and E_z have only outgoing components. The appropriate solution of (7) is then

$$E_z = e_2[\overline{\mathrm{Gi}}(\zeta - n^2) + i\overline{\mathrm{Ai}}(\zeta - n^2)], \quad (20)$$

with $e_2 = -\pi b B(0)$. Since no solution of the homogeneous form of (8) satisfies the boundary conditions, there is no ρ_0 term, i.e., (11) is replaced by $\rho = \alpha^{-1} dE_z/d\zeta$. The appropriate solution of (9) can then be written as

$$B(Z) = B_1(Z) + \tilde{B}(Z), \quad (21)$$

where $\tilde{B}$ is the same as (12). The integration by parts proceeds as in the inverse problem, giving, to lowest order, $\tilde{B}(0) = i\pi^2 q\Gamma B(0)$ and hence $B(0) = \mathrm{Ai}'(q) \times [1 - i\pi^2 q\Gamma]^{-1}$. For the electromagnetic channel, the ratio of outgoing and incident amplitudes in (21) for $Z \rightarrow \infty$ is

$$\tilde{R} = -1 + 2\pi^2 q[\mathrm{Ai}'(q)]^2 [1 - i\pi^2 q\Gamma]^{-1}, \quad (22)$$

and it is easy to show that $|\tilde{R}|^2 = |R|^2$; i.e., the reflection coefficients for the direct and inverse problem are the same. Furthermore, $|\tilde{\eta}|^2$, defined as the ratio of outgoing electrostatic energy flux, computed from (20) for large ζ, to the incident electromagnetic energy flux, computed from (21) for large Z, is just equal to $|\eta|^2$ to lowest order in β, again consistent with energy conservation. So far as the electromagnetic channel is concerned, the mode conversion appears as an "absorption" of the incident energy, so $|\tilde{\eta}|^2$ is often called the absorption coefficient. The symmetry between the direct and inverse problems is consistent with the demonstration of Means *et al.*,[9] based on time-reversal symmetry arguments.

The absorption curve, i.e., $|\eta|^2$ vs q as given by (19), is presented in Fig. 1, together with the results obtained by previous authors. There is good agreement with the numerical solutions of Forslund *et al.*[6] The results of Pert[8] and Baños and Kelly[5] are not shown, since they agree with those of Ref. 6. Piliya's[3] results, which he obtained from an analytic formula, differ significantly from all of these for reasons explained by Pert[8] and Kamp and Weenink.[10] The results of Ref. 6 shown in Fig. 1, which are for $\beta^2 = 0.015$, the smallest value cited in that work, differ by an amount of order $\beta^{2/3}$ from our curve for $q > 0.08$. This is to be expected: We have consistently assumed $\beta^2 \ll 1$, dropping terms of order $\beta^{2/3}$,

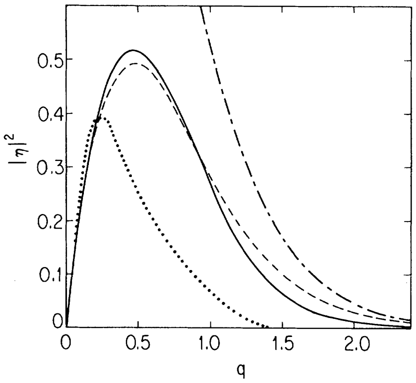

FIG. 1. Mode-conversion coefficient $|\eta|^2$ as a function of $q = (k_0 L)^{2/3} \sin^2\Theta$ as given by (19) (solid curve). Also shown are the results of the numerical solution by Forslund *et al.* (Ref. 6) (---), Piliya's (Ref. 3) analytic result (····), and the asymptotic approximation of Speziale and Catto (Ref. 7) (— — -).

and the curves of Ref. 6 show a sensitivity to β^2 above $q = 0.6$.

In summary, we have derived analytic, closed-form expressions, (16) and (19), for the reflection and mode-conversion coefficients for both the direct and inverse problems. These satisfy energy conservation and agree, to order $\beta^{2/3}$, with earlier numerical calculations of these quantities.

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