Properties of Earthquakes Generated by Fault Dynamics

J. M. Carlson and J. S. Langer
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
(Received 6 March 1989)

We present a model for fault dynamics consisting of a uniform chain of blocks and springs pulled slowly across a rough surface. The only nonlinear element of our model is a slip-stick friction force between the blocks and the surface. We find that this model gives rise to events of all sizes. Our numerical evaluation of the distribution of earthquake magnitudes results in a power-law spectrum similar to what is observed in nature. Like certain other dissipative dynamical systems, the observed large fluctuations in earthquake magnitude persist because the system is in a state of marginal stability.

PACS numbers: 91.30.Px, 05.40.+j, 05.45.+b

Systems which have no intrinsic time scales or length scales have been the subject of much investigation in recent years. While extensive work has been done to characterize certain features of these kinds of systems, until recently there has been little success at providing detailed mechanisms for their evolution. In a series of papers Bak, Tang, and Wiesenfeld[1] and, later, Kadanoff, Nagel, Wu, and Zhou[2] studied various cellular automata, loosely referred to as "sandpiles." In these models, sand is dropped slowly, one grain at a time, onto random positions on a lattice. Numerically they found that, for many different rules of evolution of the pile, the distribution of avalanche sizes displays a power-law spectrum and that, at any given time, the configuration of sand on the pile has a fractal geometry. Bak, Tang, and Wiesenfeld concluded that the observed large fluctuations in avalanche size and the spatial self-similarity result from the fact that the attractor of the dynamics is marginally stable; in essence the system is perpetually in a critical state. They introduced the term *self-organized criticality* to describe this behavior.

To the best of our knowledge, the model for earthquakes that we shall describe here is the first demonstrated example of self-organized criticality in a deterministic continuum dynamical system. This model has two main advantages compared to the cellular automata. First, because we are working with a system of differential equations, certain analytic results can be obtained. Second, there is no explicit randomness in the model; instead, the system evolves deterministically from irregular initial conditions.

The system is illustrated schematically in Fig. 1(a). Several variations of this model were studied previously both numerically and experimentally by Burridge and Knopoff.[3] The model consists of N blocks of equal mass m located at positions $X_j(t)$ along the x axis, which we imagine to be the axis of a lateral fault. Each block is connected to its two neighbors by harmonic springs of strength k_c and, when there are no additional forces on the system, the equilibrium spacing between blocks is a. The blocks are pulled individually forward through elastic couplings (leaf springs or torsion elements) of strength k_p, which move at constant loading velocity v. This loading mechanism corresponds to slow steady deformation far away from the point of contact of two tectonic plates whose linear elastic properties are described by the spring constants k_c and k_p. Finally, each block is subject to a friction force $F(\dot{X})$ which depends only on the velocity of the block. The important nonlinearities in the problem are contained in the function $F(\dot{X})$ illustrated in Fig. 1(b).

The equations of motion for this system are

$$m\ddot{X}_j = k_c(X_{j+1} - 2X_j + X_{j-1}) - k_p(X_j - vt) - F(\dot{X}_j)\,. \quad (1)$$

The rest of this Letter will be devoted to describing some solutions of these equations. Note that, unlike many traditional earthquake models, Eq. (1) contains no explicit spatial inhomogeneity[4,5] or memory effects.[6-9] Instead we will show how the dynamics alone is sufficient to generate a wide spectrum of events.

A trivial solution to Eq. (1) is given by uniform

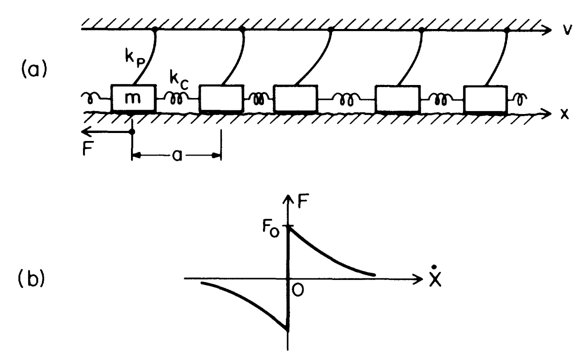

FIG. 1. (a) Block and spring system and (b) friction force $F(\dot{X})$. For the numerical results given in Figs. 3 and 4 the parameters take the values $m=1$, $k_c=2500$, $k_p=40$, $a=1$, and $v=0.01$. For the friction force we use $F(\dot{X})=F_0/(1+\dot{X})$ for $\dot{X}>0$ with $F_0=50$.

motion at the pulling velocity:

$$X_j = vt - \frac{1}{k_p} F(v) . \tag{2}$$

When the friction law is of the form illustrated in Fig. 1(b), this solution is unstable for all $v \neq 0$. A straightforward linear stability analysis of this solution reveals that all Fourier modes grow exponentially. (The growth rate remains finite at all wave numbers.) A sufficient condition for the instability is $F'(v) < 0$. In other words, because the friction decreases as the velocity increases, the system is unstable to both uniform and spatially varying perturbations in the positions of the masses.

Equation (1) also has periodic solutions which arise, for example, when the initial conditions are spatially homogeneous. In that case, translational symmetry is preserved, and the system alternately sticks (until it attains the maximum static friction) and slips (until the pulling springs are sufficiently compressed to stop the motion) in unison, as if it were a single block (see Fig. 2). Similarly, periodic solutions can take the form of kinks propagating at some speed v: $X_j(t) = X(t \pm ja/v)$.

As in the case of uniform motion, a straightforward linear stability analysis reveals that periodic solutions are unstable to spatial variations. Again this instability arises because $F'(\dot{X}) < 0$ while slipping. An important consequence of this analysis is that whenever a cluster of blocks slips together, any spatial inhomogeneity is amplified. It is this feature which leads to the interesting behavior.

Next we consider more complex situations in which

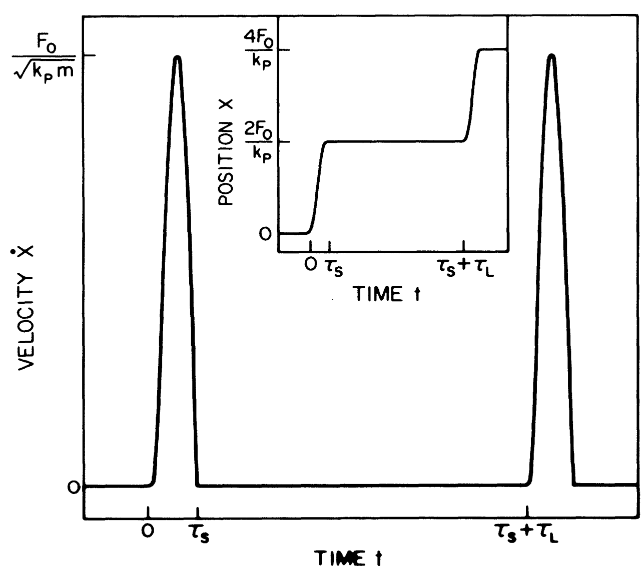

FIG. 2. Periodic solutions: velocity $\dot{X}$ and (inset) position X vs time t when the system is spatially homogeneous. The slipping time is $\tau_S = 2\pi(m/k_p)^{1/2}$, and the loading time is $\tau_L = 2F_0/k_p v$.

the system exhibits slipping events of all sizes. The following schematic description of the evolution of the system illustrates how the dynamics generates a wide range of events. Clusters of blocks which slip together do so because they happen to be in a relatively homogeneous local configuration while they are struck, and thus reach the threshold for slipping at approximately the same time. During a small slipping event, when only one block slips between two neighbors which remain stuck, the event smooths the system on the scale of three blocks (because strain cannot be released beyond the blocks that remain stuck) making a three-block slipping event more likely later at that location. On the other hand, there are two consequences of those events which involve a group of blocks slipping together between two blocks which remain stuck. First, like the single-block event, the large event smooths the system on a larger scale preparing it for subsequent larger events. Second, due to the instability associated with the friction function, any spatial inhomogeneities in the group of slipping blocks will be amplified while they are sliding. Thus, smaller events are generated persistently.

We solved (1) numerically for various different choices of the parameters, and system sizes up to 200 blocks. We considered boundary conditions in which there is zero force at the boundary (the system simply ends), and boundary conditions which damp the force at contact. Both yield similar results.[10] Generally, we started the system in a fully stuck configuration with a small spatial inhomogeneity. Before we began compiling statistics, we allowed the system to evolve for approximately ten loading periods $\tau_L = 2F_0/vk_p$, which is the maximum time it takes for the pulling strings to be stretched far enough to exceed the maximum static friction. After the system reached what appeared to be a statistically steady state, we observed a wide range of events, some of which are illustrated in Fig. 3 in the form of graphs of the velocity $\dot{X}_j$ as a function of position j and time t. The smallest events shown in Fig. 3(a) are small periodic motions involving only one block. These *creep* events never attain the pulling velocity and act to smooth the system on larger scales. Larger events, also shown in Fig. 3(a) are not periodic, and the blocks that participate in these *catastrophic* events attain velocities comparable to those shown in Fig. 2 for a uniform slipping event. Occasionally, with an average frequency roughly of order τ_L^{-1}, we see truly great earthquakes such as that shown in Fig. 3(b). In this example a localized event triggers a shock front which propagates all the way to the boundary at approximately the sound speed $a(k_c/m)^{1/2}$.

Analytically, we can characterize the two types of events—creep and catastrophic—by considering a simplified version of the model. We consider a group of n blocks in a spatially homogeneous configuration locally, and assume that this group of blocks slips as a whole between blocks on either side which happen to be stuck.

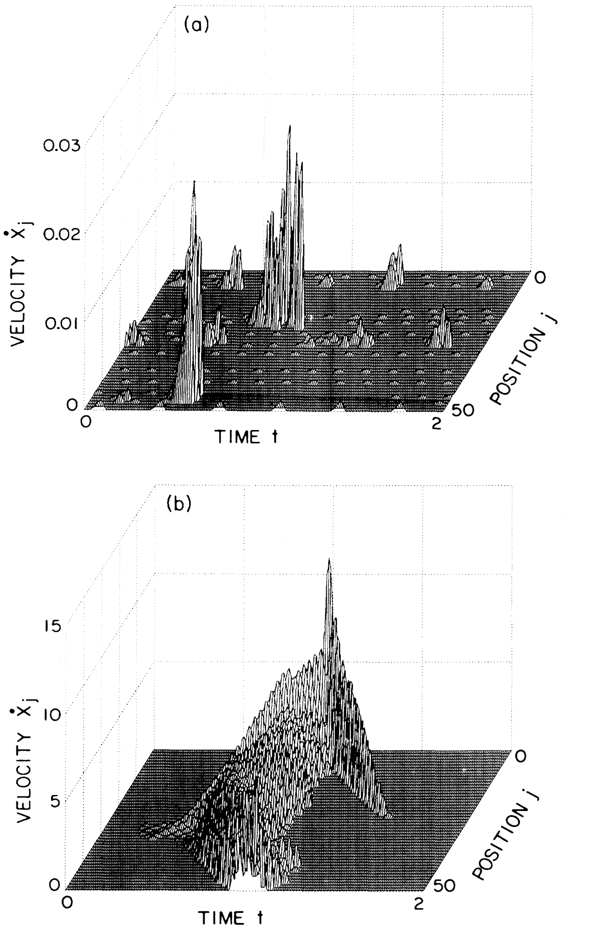

FIG. 3. Complex solutions: velocity $\dot{X}_j$ vs position j and time t. The smallest features in (a) are periodic creep events which involve only one block. The larger features are catastrophic events which involve a larger number of blocks. The great event illustrated in (b) involves all of the blocks and essentially resets the system.

The equation of motion for the slipping blocks is given by

$$nm\ddot{X} = -2k_c X - nk_p(X - vt) - nF(\dot{X}) , \qquad (3)$$

where X describes the coherent motion of the n blocks, and we have absorbed an unimportant constant on the right-hand side (which is determined by the positions of the two blocks which remain fixed) into our definition of X. Observe that all forces, except those due to the two springs which connect the slipping blocks to the rest of the system, act equally on all the slipping blocks and are thus preceded by a factor of n. In the limit of large n, (3) reduces to the equation for periodic events involving the whole system discussed previously. The important feature of the model which has been left out of Eq. (3) is the amplification of spatial irregularities while blocks slip. Our assumption in (3) is that the system is locally sufficiently smooth for an event of size n to occur. Equation (3) allows us to determine analytically the approximate amplitude and shape of an event of size n. However, in the original system [Eq. (1)], the blocks are left in a scrambled state after a slipping event, and smaller or larger events will follow.

During the small creep events, the slipping blocks attain only very small velocities, and thus to a good approximation we can linearize $F(\dot{X})$: $F(\dot{X}) \approx F_0 - F'(0)\dot{X}$. In this approximation (3) becomes the equation of motion for an antidamped harmonic oscillator with a slowly increasing bias force. Letting $X(t) \sim e^{\alpha t}$, we find that the solution oscillates (α is complex) when

$$4m\left(k_p + \frac{2k_c}{n}\right) > [F'(0)]^2 . \qquad (4)$$

This condition can be satisfied, for example, when k_c is large and n is sufficiently small. In the limit that $F'(0) = 0$ and $k_p \ll k_c$,[11] the velocity $\dot{X}$ simply oscillates sinusoidally between zero and $nk_p v/k_c$. When $F'(0) < 0$, the antidamping results in a residual force on the blocks when they restick. Because the maximum velocity of these small-amplitude events is less than the loading speed v, these events never release much of the strain accumulated in the pulling spring.

When the condition given in Eq. (4) is violated, the solution of (3) in the linearized regime grows exponentially with time (α is real). Eventually the group of blocks attains a velocity for which $F(\dot{X})$ is small. The large catastrophic events exhibit this type of behavior. We can obtain an approximate solution for $X(t)$ during a large event by letting $F(\dot{X}) \approx 0$ while the blocks are slipping. In this approximation and in the limit of slow pulling velocities, while the blocks slip (3) has a solution of the form

$$X(t) = -\frac{F_0}{\omega^2 m}\cos(\omega t) , \qquad (5)$$

where $\omega^2 = k_p/m + 2k_c/nm$, and the coefficient $F_0/\omega^2 m$ is determined by the initial conditions $\dot{X}(0) = 0$ and $-\omega^2 X(0) = F_0/m$, which specifies that the spring forces balance the threshold static friction. A more careful analysis including the linear approximation to $F(\dot{X})$ for small velocities produces similar results. In either case, unlike the small-amplitude creep events, during a large event the blocks always attain a velocity greater than the loading speed. When the blocks eventually stop slipping there is a residual force which compresses the pulling springs, resulting in a relatively long delay before these blocks are involved in another slip event.

Finally, from our numerical results we have calculated the distribution of earthquake magnitudes. The magnitude $\mathcal{M}$ is defined to be the natural logarithm of the earthquake moment M_0, which for our model can be taken to be

$$M_0 = C\sum_j \int_{\text{event}} \dot{X}_j \, dt , \qquad (6)$$

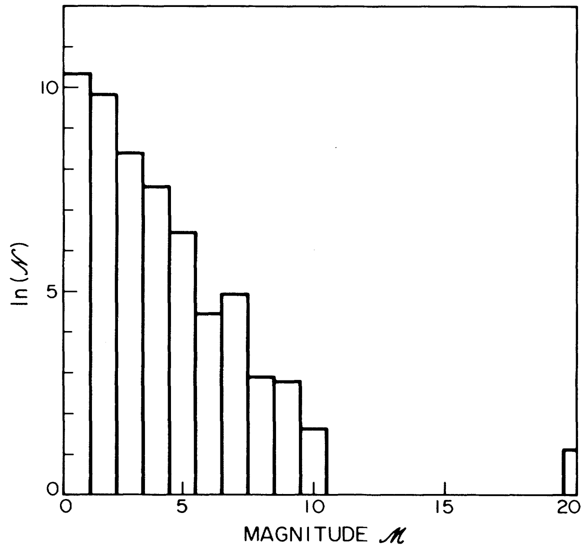

FIG. 4. Distribution of earthquake magnitudes $\mathcal{N}(\mathcal{M})$. For this result the number of blocks is $N=200$ and the constant C in Eq. (6) is $10^6 e$.

where C is a constant. The distribution $\mathcal{N}(\mathcal{M})$ produced by our numerical results is illustrated in Fig. 4, and is in strikingly good agreement with what has been observed for real earthquakes:[12] $\ln \mathcal{N}(\mathcal{M}) \approx A - B\mathcal{M}$, where A is a constant and $B \approx 1$. Even the obvious discrepancy —the frequency of great events is too large—may be consistent with observations. We tentatively attribute the latter phenomenon to the fact that both our computer model and real earthquake faults necessarily have finite lengths.

The observed large fluctuations in the magnitudes of earthquakes in nature suggest that certain dynamical models of faults may be ideal candidates for the study of general features of self-organized criticality. Many fault models incorporate an intrinsic spatial variation which is important for certain features of earthquakes. For example, Burridge and Knopoff[3] showed that local high-viscosity regions incorporated into this model can generate sequences of aftershocks in qualitatively good agreement with what is observed. In this paper we have shown that the dynamics alone is sufficient to generate a wide range of events. Somewhat alarmingly, our results suggest that small events lead to smoothing on a larger scale, resulting in larger events later on. Currently we are analyzing our results in more detail to determine the correlations between large events.

We have benefitted from many useful discussions with J. R. Rice, Chao Tang, Franco Nori, Hiizu Nakanishi, and Glen Swindle. This research was supported by U.S. Department of Energy Grant No. DE-FG03-84ER45108 and also in part by the National Science Foundation under Grant No. PHY82-17853, supplemented by funds from the National Aeronautics and Space Administration, at the University of California at Santa Barbara.

[1]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 36 (1988); C. Tang and P. Bak, Phys. Rev. Lett. **60**, 2347 (1988); J. Stat. Phys. **51**, 7977 (1988).

[2]L. Kadanoff, S. Nagel, L. Wu, and S. Zhou, Phys. Rev. A (to be published).

[3]R. Burridge and L. Knopoff, Bull. Seismol. Soc. Amer. **57**, 341 (1967).

[4]S. T. Tse and J. R. Rice, J. Geophys. Res. **91**, 9452 (1986).

[5]For related work on the theory of random pinning forces in charge-density waves, see S. N. Coppersmith and P. B. Littlewood, Phys. Rev. B **36**, 311 (1987).

[6]A. L. Ruina, J. Geophys. Res. **88**, 359 (1983).

[7]J. R. Rice, Pure Appl. Geophys. **121**, 443 (1983).

[8]F. Horowitz, Ph.D. thesis, Cornell University, Ithaca, 1989 (unpublished).

[9]J. H. Dieterich, Pure Appl. Geophys. **116**, 790 (1978).

[10]Periodic boundary conditions, however, always seem to lead to smooth propagating modes.

[11]If the system is to have sensible elastic properties in the continuum limit, $a \to 0$, then ma, $k_c a$, and k_p/a must remain finite. Thus we choose $k_c \gg k_p$.

[12]B. Gutenberg and C. F. Richter, Ann. Geofis. **9**, 1 (1956).