Superspace Geometry and Classification of Supersymmetric Extended Objects

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We provide a geometrical interpretation of the classification of supersymmetric p-dimensional extended objects. Specifically, we show that the action describing such an object exists by virtue of a nontrivial class of the $(p+2)$th Chevalley-Eilenberg cohomology of superspace, considered as the super translation group.

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The classical and quantum mechanics of extended objects is currently one of the central concerns in theoretical physics. Extended objects may arise either as particular solutions of certain field theories (e.g., cosmic strings) or they may be postulated to be fundamental (as in the 26-dimensional bosonic string theory). In the first case we can ask what will be the effective action for fluctuations of the object with wavelength long compared to its width. In this limit the extended object will behave as if it were structureless, and its effective action can depend only on its geometry. For instance, the action for a structureless p-dimensional closed bosonic extended object is[1]

$$S_D = -T\int_W d^{p+1}\xi(-\det m_{ij})^{1/2}, \tag{1}$$

where T is the p-volume "tension," $\{\xi^i, i=0,1\ldots,p\}$ are local coordinates on the "world volume" W swept out by the object in the course of its evolution from some initial to some final configuration, and m_{ij} are the components of the metric on W induced from the metric η of Minkowski spacetime M by the immersion of W into M. In local spacetime coordinates $\{X^\mu,\ \mu=0,1,\ldots,(d-1)\}$, W is defined by $X^\mu = X^\mu(\xi^i)$ and

$$m_{ij} = \partial_i X^\mu(\xi)\partial_j X^\nu(\xi)\eta_{\mu\nu}, \tag{2}$$

where the quantities $\partial_i X^\mu$ are the coordinates of the forms $\partial_i X^\mu(\xi)d\xi^i$ induced on W by the one-forms dX^μ on M through the above immersion.

The geometrical action (1) is also that of a fundamental extended object since absence of structure is precisely what is meant by the term "fundamental." At least in the case of bosonic strings, this dual role of the action (1) has long been appreciated. Only recently, however, has it been noticed that the same applies to *supersymmetric* extended objects. For example, the superstring action of Green and Schwarz,[2] originally proposed as that of a fundamental object, can be equally well interpreted as the effective action for a stringlike solution of a supersymmetric field theory.[3] This interpretation has since been explored[4,5] in the context of higher-dimensional supersymmetric extended objects,[6,7] and the results are consistent with a classification[8] of the values of p and d for which an action for a structureless supersymmetric extended object exists. One of the purposes of this paper is to present a mathematical interpretation of the results of Ref. 8.

We begin by recalling the essential elements of the construction of the supersymmetric extension of (1). Just as spacetime may be defined as the coset

$$M = \mathrm{ISO}(d-1,1)/\mathrm{SO}(d-1,1) = \text{"Poincaré"/"Lorentz,"}$$

and may be identified with the d-dimensional translation group, so superspace may be defined by

$$\Sigma = \text{"Super Poincaré"/"Lorentz,"}$$

and again the coset Σ is itself a (super)group, the super translation group. Its generators (P_μ, Q_α) obey the Lie superalgebra

$$\{Q_\alpha, Q_\beta\} = (\Gamma^\mu C)_{\alpha\beta}P_\mu, \quad [P_\mu, Q_\alpha] = [P_\mu, P_\nu] = 0, \tag{3}$$

where Γ^μ are the Dirac matrices satisfying $\{\Gamma^\mu, \Gamma^\nu\} = \eta^{\mu\nu}$, and C is the charge-conjugation matrix. The Lorentz group is an automorphism group of this algebra, with P a Lorentz vector and Q a Lorentz spinor. (For simplicity of exposition we suppose that spinors are Majorana, so that $\bar\theta = \theta^T C$; the extension of those cases for which this assumption is not applicable is straightforward.)

An important concept in this paper is that of an *invariant* differential form on a group G. If we write the group law as $g'' = g'g \equiv L_{g'}g \equiv R_g g'$, a differential form $\Pi(g)$ on G is said to be left invariant (LI) if $\Pi(L_{g'}g) = \Pi(g)$ [right invariance (RI) being defined similarly]. In our case, G will be Σ considered as the manifold of the super translation group. Let (X^μ, θ^α) be the coordinates of $g \in \Sigma$; the one-forms

$$\Pi^A = \{\Pi^\mu \equiv dX^\mu - i\bar\theta\gamma^\mu d\theta,\ \Pi^\alpha \equiv d\theta^\alpha\} \tag{4}$$

are obviously LI[9] and generate a basis for the differential forms on Σ. Given the immersion ϕ: $W \to \Sigma$ we have the induced forms $\phi^*(\Pi^\mu) = d\xi^i \Pi_i^\mu$, $\phi^*(\Pi^\alpha) = d\xi^i \partial_i\theta^\alpha$ on W. As in (2), the coefficients $\Pi_i^\mu \equiv \partial_i X^\mu - i\bar\theta\gamma^\mu\partial i\theta$ can be used to construct the induced

LI metric on W with components

$$M_{ij} = \Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu} . \quad (5)$$

The action for a *supersymmetric* structureless p-dimensional closed extended object (a "super p-brane") is then written as $S = S_1 + S_2$, where

$$S_1 = -T \int_W d^{p+1}\xi(-\det M_{ij})^{1/2} \quad (6)$$

is the obvious generalization of (1). The remaining term in the action is a type of Wess-Zumino (WZ) term, as was initially pointed out in the $p=1$ case[10] for which the action is that of the Green-Schwarz superstring. This term is best introduced by considering a $(p+2)$-form h on Σ having the following properties.

(i) h is a LI form on Σ that is also Lorentz invariant. We shall refer to this as super Poincaré invariance. It implies that h is a Lorentz scalar constructed from Π^A.

(ii) The exterior differential of h is zero, $dh=0$; i.e., h is a *closed* form. (This is, in fact, equivalent to the requirement that h also be *right* invariant under the action of the super translation group.) For the superspace Σ the closure of h implies that h is the differential of some $(p+1)$-form b,[11] $h = db$. In other words, h is *exact*, since every closed differential form on Σ is also exact (i.e., the de Rham cohomology of Σ is trivial).

(iii) If we assign dimension one to X^μ then θ^α must be assigned dimension $\frac{1}{2}$ in order that Π^μ have a definite dimension (one). With these assignments h (and hence b) should have dimension $p+1$.

Given a $(p+2)$-form h on Σ satisfying these requirements we construct S_2 as

$$S_2 = \int_W \phi^*(b) , \quad (7)$$

where, as before, $\phi^*(b)$ is the form induced on W from the form b on Σ by the map ϕ.

Requirements (i) and (ii) ensure that S_2 is a supersymmetric invariant (up to boundary terms). Requirement (iii) can be motivated as follows:[12] A *structureless* object must have a Lagrangian of a *definite* dimension, and since $(-\det M_{ij})^{1/2}$ has dimension $p+1$, we require b, and hence h, to have this dimension. Suppose that the Lagrangian were to include two terms of different dimensions; it would then involve a *relative* coupling constant of nonzero dimension. The *effective* Lagrangian would then depend on whether the scale of interest is large or small compared to the scale set by this coupling constant, thereby contradicting the assumption that the object is structureless.

The need for an S_2 satisfying requirements (i)-(iii) can also be motivated by consideration of a type of fermionic gauge invariance, generalizing that of the massless[13] and massive[14] superparticle ($p=0$), which allows half of the components of θ to be gauged away. The necessity of this "κ invariance" can in turn by motivated by stability considerations for topological defects in supersymmetric field theories.[4] A remarkable consequence of this invariance is that the gauge-fixed version of the action S_1+S_2 is a *world-volume*-supersymmetric $(p+1)$-dimensional field theory, and this may be considered as a further motivation for S_2.[3,5,8] In any case, we shall consider that S_2, and hence a $(p+2)$-form h satisfying the above requirements, is necessary for a physically acceptable action.

In this Letter we shall give a mathematical interpretation of requirements (i)-(iii) above in terms of the Chevalley-Eilenberg (CE) cohomology[15] of Σ, and explain why these requirements uniquely determine the structure of the action. It turns out that h is a Lorentz-invariant $(p+2)$-form belonging to a nontrivial class of the $(p+2)$th CE cohomology group. Some of these cohomology classes have previously played a role in the group-manifold approach to certain supergravity theories.[16] This should not be surprising since there is a close connection between supergravity theories and supersymmetric extended objects.[7,8]

We shall begin by explaining the basic concepts of CE Lie-algebra cohomology. Let G be a Lie group and $\mathcal{G}$ its Lie algebra. A p-cochain of $\mathcal{G}$ is a p-linear antisymmetric map of $\mathcal{G}\times \cdots \times\mathcal{G}$ to $\mathbb{R}$ (the general case in which $\mathbb{R}$ is replaced by a general representation space F of $\mathcal{G}$ will not be needed here). The coboundary operator δ maps the space of p-cochains $C^p(\mathcal{G},\mathbb{R})$ linearly into $C^{p+1}(\mathcal{G},\mathbb{R})$ and satisfies $\delta\delta=0$. A p-cocycle is a p-cochain c satisfying $\delta c=0$ and a p-coboundary is a "trivial" p-cocycle, i.e., one which can be expressed in terms of a $(p-1)$-cochain c' as $c=\delta c'$. Let Z^p and B^p be the vector spaces of p-cocycles and p-coboundaries, respectively. Two p-cocycles are said to be equivalent if they differ by a p-coboundary; the pth CE cohomology group $H^p(\mathcal{G},\mathbb{R})$ is then the quotient Z^p/B^p defined by this equivalence. We shall not need to specify δ because, in our case, an equivalent definition of the CE cohomology is provided by replacing the p-linear antisymmetric maps of $\mathcal{G}$ to $\mathbb{R}$ by LI forms on the group G and δ by the familiar exterior derivative d (a one-to-one correspondence between p-linear antisymmetric maps of $\mathcal{G}$ and LI differential p-forms on G can be established by means of a left translation). Then a CE p-cochain is a LI p-form on G. Let $\{e^i\}$ be a basis of LI differential one-forms on G. They satisfy the well-known Maurer-Cartan equations

$$de^i = f^i{}_{jk} e^j \wedge e^k , \quad i,j,k=1,2\ldots,\dim G , \quad (8)$$

where the $f^i{}_{jk}$ are the structure constants of $\mathcal{G}$ and the exterior product $\wedge$ of forms will be understood in the sequel. Now the p-form

$$\lambda = e^{i_1} e^{i_2} \cdots e^{i_p} C_{i_1 i_2 \cdots i_p} , \quad (9)$$

with *constant* coefficients $C_{i_1 \ldots i_p}$ is LI and therefore a p-cochain. Using (8), we see that

$$d\lambda = p f^i{}_{jk} e^j e^k e^{i_2} \cdots e^{i_p} C_{i_1 i_2 \cdots i_p} \quad (10)$$

is also a $(p+1)$-cochain. Thus, the CE cohomology group $H^p(\mathcal{G},\mathbb{R})$ is isomorphic to the cohomology group $E^p(G,\mathbb{R})$ obtained by *using the* LI *p-forms* on the group G, and the exterior derivative d. In other words, $E^p(G,\mathbb{R})$ is the de Rham cohomology group *for* LI *p-forms*: Two closed p-forms are equivalent if their difference is the differential of a $(p-1)$ LI form. Since (7) is obtained from the LI form h on $G=\Sigma$, it follows that $E^p(\Sigma,\mathbb{R})$ are *cohomology groups relevant to the superspace geometry and classification of supersymmetric extended objects.*

The extension of $E^p(G,\mathbb{R})$ to the supergroup case $G=\Sigma$ is straightforward, at least at the formal level of this Letter. A CE p-cochain of Σ is a p-form of the type

$$\Lambda=\Pi^{A_1}\cdots\Pi^{A_p}C_{A_p\cdots A_1}, \tag{11}$$

with *constant* coefficients $C_{A_p\cdots A_1}\in\mathbb{R}$. The cocycle condition is simply $d\Lambda=0$, so we can interpret the condition $dh=0$ in (ii) as the requirement that h be a CE $(p+2)$-cocycle. Suppose now that h is a trivial cocycle, i.e., a coboundary. Then $h=dc'$ for some $(p+1)$-cochain $c'\in C^{p+1}(\Sigma,\mathbb{R})$. Because c' is LI, it has to have the form [see (4)]

$$c'=\Pi^{A_1}\cdots\Pi^{A_{p+1}}C'_{A_{p+1}\cdots A_1} \tag{12}$$

for *constant* coefficients $C'_{A_{p+1}\cdots A_1}$. We shall now show that if $h=dc'$ it violates requirement (iii) or (i).

Observe that if c' consists of q factors of Π^α and $(p+1)-q$ factors of Π^μ, its dimension will be

$$\tfrac{1}{2}q+(p+1)-q=(p+1)-\tfrac{1}{2}q\,. \tag{13}$$

This equals $p+1$, as required, only if $q=0$, in which case $c'=\Pi^{\mu_1}\cdots\Pi^{\mu_{p+1}}C'_{\mu_{p+1}\cdots\mu_1}$. By requirement (i) the coefficients $C'_{\mu_{p+1}\cdots\mu_1}$ must be those of a *Lorentz-invariant* antisymmetric $(p+1)$th-rank tensor. This is possible only if $d=p+1$, i.e., a p-dimensional object moving in a $(p+1)$-dimensional spacetime. This is a degenerate case which was excluded in the analysis of Ref. 8 and which will be excluded here too.

We conclude that a $(p+2)$-form h satisfying requirements (i)-(iii) must be a CE cochain satisfying

$$dh=0,\quad h\neq dc',\quad c'\in C^{p+1}\,. \tag{14}$$

Thus, *h belongs to a nontrivial class of the $(p+2)$th CE cohomology group $E^{p+2}(\Sigma,\mathbb{R})$.*

Note that there is a b such that $h=db$ since, as we mentioned, every closed form on Σ is exact. This is nevertheless consistent with h being a nontrivial element of $E^{p+2}(\Sigma,\mathbb{R})$ because, as can be checked in those cases for which a $(p+2)$-form satisfying requirements (i)-(iii) exists, the WZ $(p+1)$-form b is *not* a CE cochain, i.e., it is not a LI form.

Let us now consider which $(p+2)$-forms h are consistent with the requirements (i) and (iii). Suppose that h consists of q factors of Π^α and $(p+2)-q$ factors of Π^μ. Then its dimension will be

$$\tfrac{1}{2}q+(p+2)-q=(p+1)+\tfrac{1}{2}(q-2)\,, \tag{15}$$

which equals $p+1$, as required, only if $q=2$. Thus the *only possible Lorentz-invariant LI $(p+2)$-form of the required dimension* is proportional to

$$h=(d\bar\theta\Gamma_{\mu_1\cdots\mu_p}d\theta)\Pi^{\mu_1}\cdots\Pi^{\mu_p}\,. \tag{16}$$

As shown in Ref. 8, requirement (ii) is then satisfied only for the values of (p,d) in one of the following four sequences:

$\mathbb{R}$:(1,3)(2,4); $\mathbb{C}$:(1,4)(2,5)(3,6);

$\mathbb{H}$:(1,6)(2,7)(3,8)(4,9)(5,10); $\mathbb{O}$:(1,10)(2,11);

associated with the four (real, complex, quaternionic, and octonionic) division algebras. We have excluded the $p=0$ case, i.e., the superparticle. In this case, the WZ term in the Lagrangian is simply $\bar\theta\dot\theta$, and can be interpreted as a mass term.[14,17] It originates from the two-form $d\bar\theta d\theta$, which is obviously closed, and is nonzero if the charge-conjugation matrix C is symmetric; otherwise, one must consider the larger $N=2$ superspace to construct a similar nonzero form. Such a two-form, and hence the mass term, determines a nontrivial element of $E^2(\Sigma,\mathbb{R})$. As is well known, a nontrivial second cohomology group of a Lie algebra characterizes the extension of the algebra by an Abelian one. In the case of the supersymmetry algebra this is nothing other than the familiar extension by a central charge. [Also, superspace itself, viewed as a central extension group, is the result of the existence of the closed two-form $d\bar\theta\Gamma^\mu d\theta$ (Ref. 18)]. Is there an analogous physical significance to the higher forms of the CE cohomology groups?

In the group-manifold approach to supergravity the higher cohomology groups are related to extensions of the supersymmetry algebra to "free differential algebras," or "Cartan integrable systems."[16] Also, higher-order WZ terms play an essential role in the theory of anomalies (see, e.g., Ref. 19 and references therein). Our work suggests that the differential forms of the higher CE cohomology groups are related to extensions of the supersymmetric *current* algebra. It is known that the world-volume current algebra of supersymmetric extended objects *cannot* be that of ordinary supersymmetric field theory,[3] but must contain an additional term. This is because the action of the extended object, considered as a $(p+1)$-dimensional field theory, exhibits a *partial* breaking of rigid supersymmetry (PBRS) and this is possible only in the presence of the additional term. It has been conjectured[5] that this extension of the usual current algebra is possible only for those cases tabulated above. In this case, the extensions of the supersymmetry current algebra of this type are possible only when there is an appropriate nontrivial CE higher-order cohomology group of Σ. The relation between a nontrivial element of a CE cohomology group and the additional central term in the algebra of supercurrents for models exhibiting PBRS will be presented elsewhere.

To summarize, the only $(p+2)$-differential forms h on Σ satisfying our requirements are those of the form (16) and then only for the values of (p,d) tabulated above. We have shown that each such form is a nontrivial element of the CE cohomology group $E^{p+2}(\Sigma,\mathbb{R})$. It should be stressed, however, that there are other nontrivial elements of the CE cohomology group for which conditions (i) and (ii) are satisfied, but which do not satisfy condition (iii). An example is the closed four-form $(d\bar{\theta}\Gamma_{\mu\nu}d\theta)(d\bar{\theta}\Gamma^{\mu\nu}d\theta)$ of dimension two, for an (11,32)-dimensional superspace. An interesting question is whether these other nontrivial elements of the CE cohomology groups of Σ also have a physical interpretation.

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