Time of Zener Tunneling

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(Received 22 February 1989)

The time of Zener tunneling, τ_Z, is calculated for a general two-level quantum-mechanical system by two methods. In the first we determine the width of the transition profile in time. In the second we apply an oscillating perturbation and examine how the final transition probability depends upon the perturbation frequency and phase. Both methods show, given that the coupling energy between the diabatic levels is Δ and the time rate of change of the energy is α, that in the adiabatic limit τ_Z scales as Δ/α and in the sudden limit τ_Z scales as $(\hbar/\alpha)^{1/2}$. We calculate the approximate transition probability at finite times for the two limits. The results can be applied to specific mesoscopic electronic systems.

PACS numbers: 73.40.Gk, 03.65.—w

Since the problem of level crossing (Zener tunneling[1-6]) was first addressed circa 1930, its application to many fields has been recognized. Examples include atomic collisions, atom-surface scattering, molecular physics,[7] and molecular biology.[8-10] More recently this problem has been introduced into the growing field of submicron physics. Two apt examples are recent studies of ultrasmall tunnel junctions driven by an external current source,[11-17] and the response of small normal-metal rings to a time-dependent Aharonov-Bohm flux.[18] In these and many other cases one deals with level diagrams involving multiple level crossings. This is illustrated in Fig. 1, in which the instantaneous energy eigenvalues of a system are plotted as function of ϕ, the external parameter (magnetic flux, electric charge, the distance of the atom from the surface, etc.). Although the details of the microband structure depend on the specifics of the problem,[19,20] their qualitative features are quite general.

When ϕ varies with time, one often has to deal with consecutive Zener transitions. In the present work we point out that in order to facilitate such an analysis, it is necessary to characterize these Zener transitions not only in terms of asymptotic probabilities (or probability amplitudes), but also by the time it takes the system to Zener tunnel. This Zener time, τ_Z, determines whether consecutive Zener events can be considered as separable. Also, by comparing τ_Z to another time scale in the problem—the phase-coherence time τ_ϕ—it is possible to determine whether consecutive Zener tunneling events should be treated as sequential and incoherent, described, e.g., by a master equation, or as sequential, coherent events.

We have conducted a systematic study of Zener time, using several alternative approaches, both analytic and numerical. We have found two different limiting expressions for τ_Z in terms of the parameters of the problem. In the almost *adiabatic limit* we find

$$\tau_Z \approx \Delta/\alpha\,, \tag{1}$$

where 2Δ is the narrow gap separating two consecutive microbands (cf. Fig. 1) and α is the rate at which energy is exchanged with the external source in the limit Δ is zero ($\alpha \equiv \lim_{\Delta\to 0} dE/dt$). Equation (1) is valid for $\gamma \equiv \hbar\alpha/\Delta^2 \ll 1$. In the *sudden* limit ($\gamma \gg 1$) we find

$$\tau_Z \approx (\hbar/\alpha)^{1/2}. \tag{2}$$

The result of Eq. (1) has been suggested earlier on the basis of heuristic arguments[9] but its limited validity ($\gamma \ll 1$) had not been discerned. Büttiker and Landauer

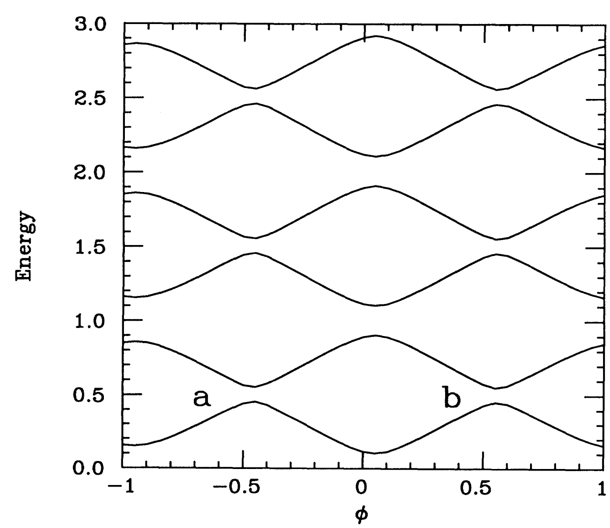

FIG. 1. A plot of the instantaneous energy eigenstates of a system with multiple level crossings, as a function of some external control parameter ϕ. The points labeled a and b are gaps between the two lowest bands referred to in the text.

have derived a result equivalent to Eq. (1) for Zener breakdown (tunneling in k space).[21] The result of Eq. (2), as well as the distinction between τ_Z in the sudden and adiabatic limits, is new. We find that our definitions of the Zener time derived using both an "internal clock" and an "external clock" (each carrying a different physical meaning) yield compatible results for τ_Z. This differs from tunneling in real space in which different definitions of the time of tunneling which rely on either an external or an internal clock may yield different results. We also derive results, pertaining to multiple Zener dynamics, for the tunneling over finite time, as opposed to the asymptotic probability. Our analytic results have been confirmed by extensive numerical studies. Below we give some details of our analysis. We then conclude by commenting on possible physical implications.

To define the time of Zener tunneling we consider a driven two-level system described by the Hamiltonian

$$H_0 = \alpha t S_z + \Delta S_x \,, \tag{3}$$

where S_x and S_z are Pauli spin-$\frac{1}{2}$ matrices. Here α is the rate of change of the energy of the uncoupled ($\Delta=0$) levels due to the external driving bias and Δ is the coupling between the levels.

It is convenient to define the state of the system $\psi(t) = a_+(t)\,|+\rangle + a_-(t)\,|-\rangle$ in terms of the *diabatic* states which satisfy $S_z\,|\pm\rangle = \pm\,|\pm\rangle$. We chose as an initial condition $a_+(-\infty)=1$ and $a_-(-\infty)=0$. The asymptotic Zener probability is[1-6]

$$P_+(\infty) = |a_+(+\infty)|^2 = e^{-\pi/\gamma}. \tag{4}$$

To define the time for Zener tunneling we employ two main approaches. The first one (the internal clock) is to investigate the probability profile $P_+(t) \equiv |a_+(t)|^2$ (Fig. 2). We can identify the crossover time from the initial probability $P_+(-\infty)$ to the asymptotic probability $P_+(+\infty)$ as the transition time and study how it scales with α and Δ. The second approach (external clock) is to add an oscillating component to H_0 and identify the inverse Zener time as a characteristic frequency appearing in the response of the system to the perturbation.[22]

Width of the probability profile.— In the sudden limit we can expand Eq. (4) in powers of γ^{-1}. We therefore look for a solution of the Schrödinger equation of the form $a_+(t) = \sum \gamma^{-n/2} a_{+,n}(t)$, with an initial condition $a_+(t_0)=1$ and $a_-(t_0)=0$, where t_0 is a large, negative time. In terms of $y \equiv t/(\hbar/\alpha)^{1/2}$, we obtain to order γ^{-1},

$$a_+(y) \approx e^{-i(y^2-y_0^2)/2}\left\{1 - \frac{1}{\gamma}\int_{y_0}^{y} e^{iu^2}\int_{y_0}^{u} e^{-iv^2}\,du\,dv\right\}. \tag{5}$$

Using Eqs. (4) and (5) one can see that the rescaled profile $[P_+(t)-P_+(\infty)]/[1-P_+(\infty)]$ is a "universal" curve for $\gamma \gg 1$. In particular we have shown that the width of the transition scales as $(\hbar/\alpha)^{1/2}$.

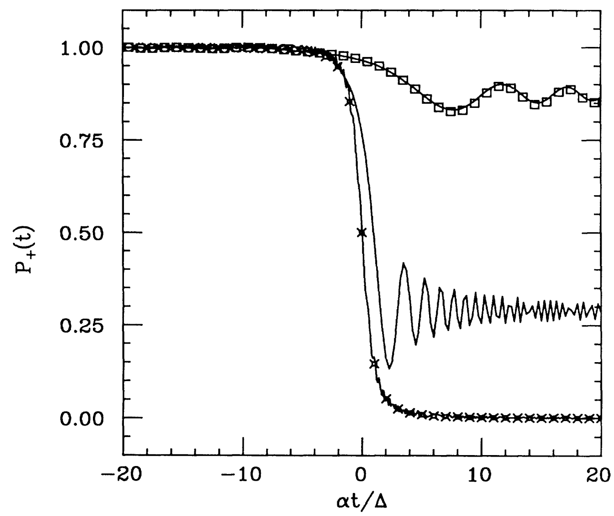

FIG. 2. A plot of $P_+(t)$, the probability of being in state $|+\rangle$ at time t. The parameters are $\gamma=20$ (top), $\gamma=2.5$ (middle), and $\gamma=0.2$ (bottom). One definition of Zener time is the width of the transition from $P_+ \approx 1$ to its asymptotic value, $P_+ = \exp(-\pi/\gamma)$. The symbols plotted are points calculated using the analytic approximations for the profiles, given in the text.

In the adiabatic limit we assume a WKB form for $a_+(t)$. In terms of the dimensionless time $x \equiv \alpha t/\Delta$ we write $a_+(x) \approx K(x)\exp[iS_0(x)/\gamma]$. Substituting in the Schrödinger equation and equating different orders of γ we obtain

$$S_0(x)_\pm = \pm\tfrac{1}{2}\{\ln[x+(1+x^2)^{1/2}] + x(1+x^2)^{1/2}\}\,, \tag{6a}$$

$$K(x)_\pm = (1+x^2)^{-1/4}[x+(1+x^2)^{1/2}]^{\mp 1/2}. \tag{6b}$$

The solution $a_+ \approx K(x)_+\exp[iS_0(x)_+/\gamma]/\sqrt{2}$ obeys the boundary condition $a_+(-\infty)=1$. Since the solution has an invariant profile when time is measured in units of Δ/α, we conclude that the width of the transition and thus the Zener time scales as Δ/α in the adiabatic limit.[23]

Both of the above analytic results are in good agreement with numerical calculations of $a_+(t)$, as shown in Fig. 2.

Sensitivity to ac perturbation.— An alternative method, motivated by the Büttiker-Landauer calculation of tunneling in real space,[24] consists in introducing an external ac perturbation in the Hamiltonian and looking for some characteristic frequency in the response of the system to the perturbation. Unlike Ref. 24 we identify τ_Z by employing the concept of a phase shift, θ. We first consider the sudden limit. We add a perturbation to H_0 of the form $H_1 = (\epsilon/2)\sin(\omega t+\theta)S_x$. For $\omega\tau_Z \ll 1$ we expect $P_\pm(\infty)$ to depend weakly on ω but to vary with θ due to the static shift in the coupling: $\tilde{\Delta} = \Delta + \epsilon/2\sin\theta$. For $\omega\tau_Z \gtrsim 1$ we expect $P_\pm(\infty)$ to depend on ω. We determine τ_Z as the inverse of the characteristic frequency at which the crossover between the two behaviors occurs. We write the amplitudes

$$a_{\pm}(t)=\tilde{a}_{\pm}(t)\exp\left[(1/i\hbar)\int^{t}E_{\pm}(t')dt'\right]$$

with $E_{\pm}\equiv\langle\pm|H_0+H_1|\pm\rangle$. From the Schrödinger equation we have

$$\dot{\tilde{a}}_{-}(t)=\langle -|H|+\rangle\exp\left[(i/\hbar)\int^{t}(E_{+}-E_{-})dt'\right].$$

Approximating $\tilde{a}(t)\approx 1$ [recall that $P_{-}(\infty)\ll 1$] we finally obtain

$$P_{-}(\infty)\approx\frac{\pi}{\gamma}\left[1+\frac{\epsilon}{\Delta}\sin\theta\cos\frac{\hbar\omega^2}{4\alpha}+\frac{\epsilon^2}{\Delta^2}\sin^2\theta\right],\quad (7)$$

which agrees with direct numerical calculations (Fig. 3).[25] We identify a characteristic frequency $(\alpha/\hbar)^{1/2}\approx\tau_Z^{-1}$. For $\omega\tau_Z\ll 1$ we see that $P_{\pm}(\infty)$ approaches a constant, which can be obtained directly by inserting $\tilde{\Delta}$ into Eq. (4).

We have also used the above approach to study a different ac perturbation, $H_2=(\epsilon/2)\sin(\omega t+\theta)S_z$. To leading order in $\eta\equiv\epsilon/\hbar\omega$ we obtain

$$P_{-}(\infty)\approx\frac{\pi}{\gamma}\left[1-2\eta\cos\theta\sin\frac{\hbar\omega^2}{4\alpha}\right],\quad (8)$$

which also agrees with our numerical results (Fig. 3). Again, for $\omega\Delta/\alpha\ll 1$ the relevant time scale is $\approx(\hbar/\alpha)^{1/2}$.

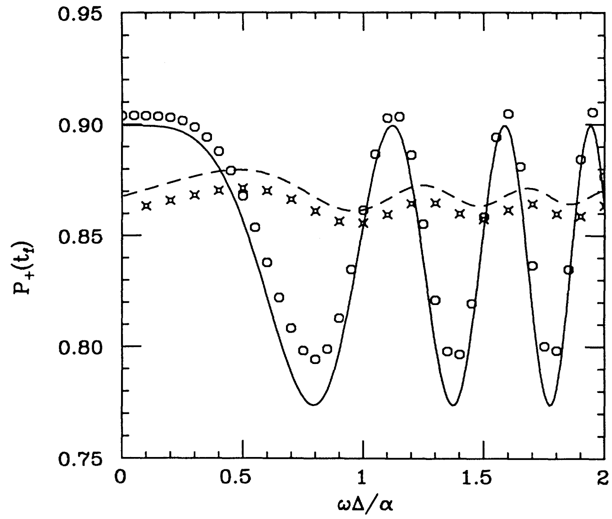

FIG. 3. A plot of the asymptotic Zener tunneling probability as a function of the frequency of the perturbation in the sudden limit. The solid line is the analytic expression [Eq. (7)] derived for the perturbation $H_1=(\epsilon/2)\sin(\omega t+\theta)S_x$, and the dashed line is the analytic expression [Eq. (8)] for the perturbation $H_2=(\epsilon/2)\sin(\omega t+\theta)S_z$. The parameter γ is 20. The plotted symbols are the results of numerical solutions of the time-dependent Schrödinger equation for the Hamiltonian of Eq. (3).

We now turn to the adiabatic limit. We discuss a particular ac perturbation that does not modify the adiabatic eigenstates of the system:

$$H_3=(\epsilon H_0/2)\sin(\omega t+\theta)/(\alpha^2t^2+\Delta^2)^{1/2}.$$

We employ the standard adiabatic approximation

$$a_{\pm}(t)=-\frac{\alpha\Delta}{2}\int_{-\infty}^{\infty}\frac{dt'}{\alpha^2t'^2+\Delta^2}\exp\left[\frac{2i}{\hbar}\int^{t'}[\alpha^2t'^2+\Delta^2+\epsilon\sin(\omega t'+\theta)]dt'\right].\quad (9)$$

When ϵ is sufficiently small[26] we expand in η and evaluate the integral using the method of steepest descent. For $\omega\Delta/\alpha$ we obtain

$$P_{+}(\infty)\approx\frac{\pi^2}{9}\exp\left[\frac{\pi}{\gamma}-2\eta\sinh\frac{\omega\Delta}{\alpha}\sin\theta\right],\quad (10)$$

which, up to a multiplicative factor, agrees with our numerical results.[27] For $\omega^{-1}\gg\Delta/\alpha$ we find that $P_{+}(\infty)$ does not depend upon ω; the perturbation acts as a static shift in Δ. For larger values of ω the system undergoes many oscillations while tunneling takes place, and the sensitivity to θ vanishes. The crossover from the small-ω limit to the large-ω regime defines τ_Z, in accordance with Eq. (1).

Finally, we comment on some physical applications of our work. Consider Zener tunneling events between the first and second band in the vicinity of the narrow gaps a and b in Fig. 1. It has been previously assumed[28-30] that the complex dynamics of such systems can be decoupled into a series of separate two-level Zener events, each described by a Hamiltonian of the form given in Eq. (3) and a corresponding tunneling probability [Eq. (4)]. To examine the above assumption it is useful to define another important time scale: τ_p, the time interval between two narrow gaps. (For example, in one-dimensional rings threaded by an Aharonov-Bohm flux $\tau_p=\phi_0/\alpha$.) If $\tau_Z\gtrsim\tau_p$, consecutive Zener events cannot be decoupled. In this case a full quantum-mechanical treatment allowing interference between the amplitudes in each band must be pursued. Moreover, for $\tau_Z>\tau_\phi$, the probability of a single event is not given by Eq. (4); the latter assumes phase coherence over a time greater than τ_Z. Our analysis allows us to calculate Zener transitions which take place over a finite time interval.

For an Aharonov-Bohm ring of perimeter ≈ 6000 Å, a cross section of 300×300 Å^2, and a resistance of 10 kΩ the decoupling assumption of consecutive Zener transitions breaks down for a rate of change of the magnetic field $dB/dt>10^6$ G/sec. For a small Josephson junction with a charging energy on the order of its Josephson coupling energy ≈ 0.5 K, this happens for an external driving current $I>10^{-8}$ A.

The perturbations H_1 and H_2 above can be realized explicitly in experiments. In the case of an ultrasmall capacitance Josephson junction driven by an external current source then the application of an additional, oscillating current source[31] would serve as a perturbation of the form H_1, while an oscillating external magnetic field would alter the strength of the Josephson coupling and thereby act as a perturbation of the form H_2. Similarly, in the case of Aharonov-Bohm rings, the application of an additional oscillating magnetic flux would serve as a perturbation of the form H_1. The determination of the Zener tunneling probability as a function of ω would allow a direct comparison between theory and experiment.

We thank R. Landauer for useful discussions and collaboration in the early stages of this project. We also thank S. Fishmann, A. Stern, and D. J. Thouless for their comments. K. M. and E. B.-J. would like to thank the Weizmann Institute, and Y. G. the University of Michigan, for hospitality during our respective visits. This research was partially supported by NSF Grant No. DMR 8608305, Grant No. DAAL 03-87-k-0007 from the Army Research Office, the U.S.-Israel Binational Science Foundation (Jerusalem), the Minerva Foundation (Munich, Federal Republic of Germany), and the Presidential Initiative Fund Kellog Grant at the University of Michigan. Computer simulations on the San Diego Super Computer were supported by the NSF. One of us (K.M.) was supported by the Center for High Frequency Micro-electronics at the University of Michigan.

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[22]The terms "internal" and "external" might seem confusing since the former still depends upon the action of the external dc driving force. An alternative description would be "inherent" and "applied," respectively.

[23]We note parenthetically that this approximate solution is good only for $x < \gamma^{-1}$, and does not give the correct result for the probability of Zener tunneling at $x \rightarrow +\infty$. Such a perturbation calculation cannot determine the exponentially small Zener transition probability.

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[26]More specifically, this expansion is justified for $\omega\Delta/\alpha < 1$, $\eta < 1$, and $\frac{9}{32}[\epsilon \cosh(\omega\Delta/\alpha)\sin\theta/\Delta]^3 \ll 1$.

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