Random-Field Mechanism in Random-Bond Multicritical Systems

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It is argued on general grounds that bond randomness drastically alters multicritical phase diagrams via a random-field mechanism. For example, tricritical points and critical end points are entirely eliminated ($d \leq 2$) or depressed in temperature ($d > 2$). These predictions are confirmed by a renormalization-group calculation. Another consequence of this phenomenon is that, under bond randomness, the phase transitions of q-state Potts models are second order for all q at dimensionality $d \leq 2$.

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Quenched randomness affects phase-transition phenomena in varying degrees. For example, bond randomness changes the numerical values of the critical exponents, and then only if the specific-heat exponent α is positive in the nonrandom system.[1] On the other hand, field randomness can altogether eliminate the phase transitions of low-dimensional systems[2-4] and, otherwise, always affects the numerical values of the critical exponents.[5] Thus, the effect of bond randomness has to date been thought of as less drastic than that of field randomness.

Nevertheless, we predict here that bond randomness does in fact drastically affect multicritical phase diagrams.[6] We identify a random-field mechanism that is operative in random-bond multicriticality. Accordingly, both tricritical points and critical end points are entirely eliminated by bond randomness, in two dimensions and less ($d \leq 2$). Another consequence is that q-state Potts models have second-order phase transitions for all q in $d \leq 2$ under random bonds, in contrast to the first-order behavior occurring for $q > q_c$ in pure systems (e.g., $q_c = 4$ for $d = 2$). In general, in all dimensions, first-order behavior cedes to some degree to second-order behavior upon introduction of bond randomness.

We follow here the actual sequence of the reported research. We begin with an *a priori* intuitive and general argument that unveils this random-field mechanism that is operative in the absence of random fields. We follow by an actual renormalization-group calculation that confirms the effects. It is demonstrated that tricritical and critical-end-point phase diagrams are replaced by purely second-order phase diagrams, which also exhibit reentrance phenomena.[7] Finally, other consequences of this mechanism are given, including the major shift in q_c in Potts models and the effect on more exotic multicritical phase diagrams.

The basic multicritical phase diagrams are found in the Blume-Emery-Griffiths (spin-1 Ising) model,[8] with the Hamiltonian

$$-\beta\mathcal{H} = J\sum_{\langle ij\rangle} s_i s_j + K\sum_{\langle ij\rangle} s_i^2 s_j^2 - \Delta\sum_i s_i^2 ,$$

where $s_i = 0, \pm 1$, at each site i of a lattice, and $\langle ij\rangle$ indicates summation over all nearest-neighbor pairs of sites. We consider attractive interactions ($J, K \geq 0$). Note the two limiting problems in this model.

(1) For $\Delta = -\infty$, $s_i = \pm 1$ at all sites, and the model reduces to the spin-$\frac{1}{2}$ Ising model. A second-order phase transition occurs at $J = J_c$ (e.g., $J_c = 0.4407$ and 0.2216 for the square and cubic lattices). When Δ is increased from $-\infty$, some sites have $s_i = 0$. This weakens the connectivity of the magnetic interactions, so that we expect a gradual depression of the transition temperature given by $1/J_c$. Thus, a line of second-order transitions will drop from the point $[\Delta = -\infty,\ 1/J_c(-\infty)]$, as seen on the left-hand side of Fig. 1(a).

(2) As Δ/J is scanned at low temperatures ($J, \Delta \to -\infty$), the system will exhibit a simple problem of phase separation between the magnetic ($s_i = \pm 1$) and

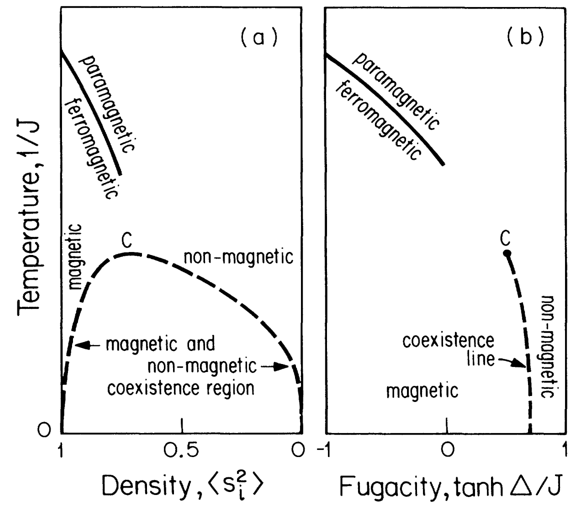

FIG. 1. A multicritical phase diagram is the merger of an order-disorder phase transition at high density (solid lines) and a dense-dilute phase separation at low temperature (dashed lines). Tricritical or critical-end-point phase diagrams result when the order-disorder transition line reaches the top or the side, respectively, of the magnetic-nonmagnetic coexistence region.

nonmagnetic ($s_i = 0$) states. Thus, a coexistence region occurs, as seen in the lower portion of Fig. 1(a).

Multicriticality is the merger of the two pieces of the problem deduced above. If the second-order line hits the very top of the coexistence region, a tricritical point occurs. This happens for $K/J < R$ in the Blume-Emery-Griffiths model, where $R = 3$ for $d = 2$ and $R > 3$ for $d > 2$.[9] On the other hand, if the second-order line hits the side of the coexistence region, a critical end point occurs, as it happens for $K/J > R$ in the Blume-Emery-Griffiths model.

Consider now any case of random bonds. One example is a quenched distribution $\mathcal{P}(J)$ of bond strengths J. Under scale change (renormalization-group transformation), this will correspond to a quenched distribution $\mathcal{P}(J,K,\Delta)$ of all even (spin-reversal invariant) interactions in the thermodynamically equivalent rescaled system. Consider the isolated point C in Fig. 1(b). This is the critical point of the limiting problem (2) discussed above. The randomness $\mathcal{P}(J,K,\Delta)$ mixes points in the coexistence direction subtending C [the direction of the dashed line in Fig. 1(b)] with points outside the coexistence direction. In other words, the randomness has components orthorgonal to the coexistence boundary. Thus, the randomness $\mathcal{P}(J,K,\Delta)$ acts, on C and its subtending coexistence line, just like field randomness.

In more physical words, Δ is the chemical-potential difference between magnetic and nonmagnetic states. The value of Δ in the coarse-grained (renormalized) system clearly depends on the value of the bond strength J in the underlying more microscopic (unrenormalized) system. Thus randomness in J induces under scale change (renormalization) randomness in Δ. Finally, the chemical-potential difference Δ acts as a field on the phase separation critical point C.

Accordingly, C is depressed in all dimensions. It is depressed to zero temperature, and entirely eliminated with its subtending coexistence line, in $d \le d_l$. Since C with its subtending coexistence line is a phenomenon of magnetic-nonmagnetic phase separation, also known as dense-dilute phase separation, it is in the universality class of the ($n = 1$)-component spin systems, with $d_l = 2$.[3,4] Thus, component (2) of multicriticality, as analyzed above, disappears for $d \le 2$. Component (1), the second-order line, reaches zero temperature. The multicritical phase diagram, whether tricritical or critical end point, is eliminated.

If we think of the global phase diagram in the (J,K,Δ) space,[9] we realize that C is in fact part of an isolated critical line, parametrized by K/J and subtended by a surface of coexistence. The randomness $\mathcal{P}(J,K,\Delta)$ of course still has components orthogonal to the coexistence surface, and the above reasoning holds.

In the global phase diagram in (J,K,Δ) space, the second-order line of component (1) above is in fact part of a second-order surface parametrized by K/J. It is subtended by a region of coexistence with the full dimensionality of all even interactions. Thus, the randomness $\mathcal{P}(J,K,\Delta)$ does not have components orthogonal to this coexistence locus. This is why no random-field mechanism is operative on component (1) of would-be multicriticality. Thus, component (1) survives bond randomness.

We have checked this *a priori* argument by performing an actual renormalization-group calculation. The recursion of the quenched probability distribution under rescaling is[10,11]

$$\mathcal{P}'(\mathbf{V}') = \int \left[\prod_{\langle ij \rangle}^{b^d} d\mathbf{V}_{ij}\, \mathcal{P}(\mathbf{V}_{ij}) \right] \delta(\mathbf{V}' - \mathbf{R}(\{\mathbf{V}_{ij}\}))\,,$$

where $\mathbf{V}_{ij} \equiv (J_{ij}, K_{ij}, \Delta_i, \Delta_j)$, $\mathbf{R}$ is a local recursion relation, b is the length rescaling factor, and primes denote the rescaled system. For the local recursion relation, we have used the Migdal-Kadanoff approximation in its locally differentiating form, which correctly produces the lower-critical dimension of the random-field problem.[11] Here, it yields four local renormalized interactions from $4b^d$ different local unrenormalized interactions as arguments. We have parametrized $\mathcal{P}(\mathbf{V})$ with sixteen histograms.[10,11] From the study of the fixed points and global renormalization-group flows, we have determined phase diagrams. Densities have been calculated via the approximate recursion formula[12]

$$\mathbf{n} = b^{-d}\mathbf{n}' \cdot \mathbf{T}\,,$$

where

$$\mathbf{T} = \int \left[\prod_{\langle ij \rangle}^{b^d} d\mathbf{V}_{ij}\, \mathcal{P}(\mathbf{V}_{ij}) \right] \sum_{\langle ij \rangle}^{b^d} \frac{\partial \mathbf{Q}}{\partial \mathbf{U}_{ij}}\,,$$

where $\mathbf{Q}$ and $\mathbf{U}$ are $\mathbf{R}$ and $\mathbf{V}$ augmented by the additive

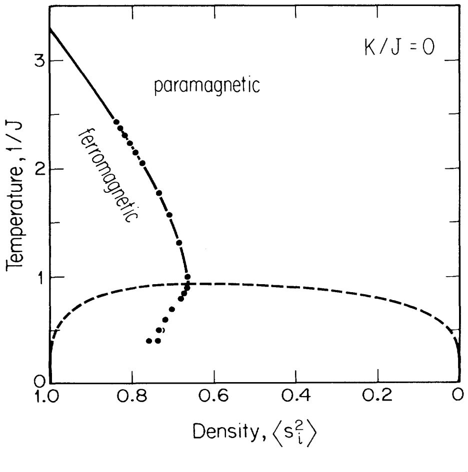

FIG. 2. Effect of bond randomness on a tricritical phase diagram. The curves are for the pure system. The solid circles are the phase-boundary points for the random-bond system with $\mathcal{P}(J_{ij}) = [\delta(J-\sigma) + \delta(J+\sigma)]/2$, $\sigma = J/4$. The lowest phase-boundary point is split, because of the more than 14-digit accuracy in chemical potential needed to reach the phase boundary from either side [see Fig. 3(b)].

constant per bond $\langle ij\rangle$, and $\mathbf{n}$ is the conjugate densities.

The curves in Fig. 2 show a tricritical phase diagram of the pure system in $d=2$, calculated as described above. When the bilinear coupling J (see the first equation of the paper) is randomized according to the quenched probability distribution $\mathcal{P}(J_{ij})$ (which constitutes the initial point of the renormalization-group trajectories), the phase-boundary points given by the solid circles are obtained. As predicted, the tricritical point and the ferromagnetic-nonmagnetic coexistence region are eliminated. We have checked, in temperature-chemical-potential space, that the second-order phase boundary reaches zero temperature. Furthermore, as a surprise result, Fig. 2 shows reentrant behavior: For a range of densities, as temperature is lowered, the system first passes from the paramagnetic phase to the ferromagnetic phase, then from the ferromagnetic phase back to the paramagnetic phase.

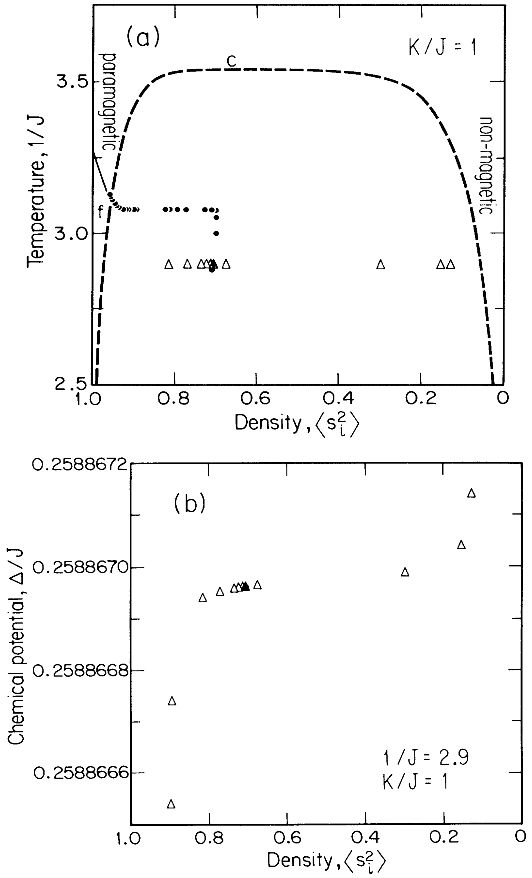

FIG. 3. (a) Effect of bond randomness on a critical-end-point phase diagram. The curves are for the pure system. The solid circles are the phase-boundary points for the random-bond system with $\mathcal{P}(J_{ij})=[\delta(J-\sigma)+\delta(J+\sigma)]/2$, $\sigma=J/4$. The triangles show our determination of one of these points, by successive bisection of a chemical-potential interval straddling the boundary. These triangles explicitly show homogeneous-phase existence in what was coexistence region in the pure system. (b) Calculated isotherm of the random-bond system, corresponding to the bisections exhibited in (a). The solid circle gives the phase-boundary point. In the pure system, this isotherm has a density jump from $\langle s_i^2\rangle=0.971$ to 0.058.

Previous renormalization-group work[13] on the $d=2$ random-bond Blume-Emery-Griffiths model had seen a depression of the tricritical temperature. However, the full effect, namely the total elimination of the tricritical point, was not seen because the randomness generated for the renormalized Δ was approximated away in the previous work.

The curves in Fig. 3(a) show a calculated critical-end-point phase diagram of the pure system in $d=2$. Again, when bond randomness is introduced via the quenched probability distribution $\mathcal{P}(J_{ij})$, the phase-boundary points given by the solid circles are obtained. The triangles show our determination of one of these points, by successive bisections of a chemical-potential interval straddling the boundary. These triangles explicitly show homogeneous-phase existence in what was a coexistence region in the pure system. Under bond randomness, the critical end point, the isolated critical point, the ferromagnetic-nonmagnetic coexistence region, and the paramagnetic-nonmagnetic coexistence region area all eliminated. We have again checked that, in temperature–chemical-potential space, the second-order phase boundary reaches zero temperature.

We have repeated the same calculations for $d=2.58$ and $d=3$. Although depressed in temperature, the tricritical point and the critical end point survive bond randomness in both $d>2$ systems, as predicted.

The argument given in this paper clearly applies to all tricritical and critical-end-point phase diagrams, including discrete or continuum n-component spin systems, antiferromagnets and metamagnets, etc. Thus, the lower-critical dimension d_l for random-bond tricriticality and end-pont criticality is 2. As d is increased from $d_l=2$, tricritical points and critical end points emerge from zero temperature.

The application to Potts models has further-reaching significance. The renormalization-group theory of q-state Potts models encompasses the extended space of the

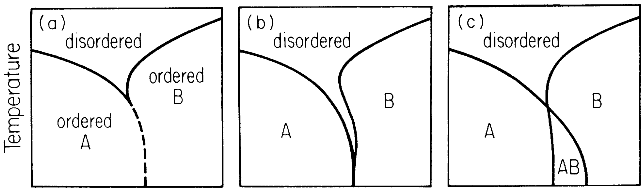

FIG. 4. Bond randomness should convert a bicritical phase diagram (a) into a disorder-line phase diagram (b) or a decoupled-tetracritical phase diagram (c), by the random-field mechanism discussed in this paper.

Potts lattice-gas model.[14] In this space, as q is increased, critical and tricritical fixed points merge and annihilate at q_c, leaving way to first-order transitions for $q > q_c$.[15] In the present case, in $d \leq 2$ random bonds eliminate the tricritical fixed point, so that the merger and annihilation cannot occur. Again, the random bonds act as random fields (random chemical potentials) on the effective vacancies that would otherwise cause the first-order transitions by their condensation.[16] Thus, in $d = 2$, although the pure Potts models have first-order transitions for all $q > q_c = 4$, the random-bond Potts models have second-order transitions for all q.

This random-field mechanism of random bonds is also applicable to more exotic multicritical phase diagrams. For example, Fig. 4 shows how a bicritical phase diagram is affected.

Finally, we note that in a parallel work, Aizenman and Wehr[17] have rigorously shown that the q-state random-bond Potts models have no latent heat in $d \leq 2$. Their work and ours were completed independently and are supplementary to each other: Reference 17 is cast in the general context of the vanishing of the discontinuities in the thermodynamic densities; our work draws on phenomenological arguments and approximate renormalization-group calculations to infer the second-order nature of the transition.

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