Rounding of First-Order Phase Transitions in Systems with Quenched Disorder

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In 2D, quenched randomness results, quite generally, in the elimination of discontinuities in the density of the variable conjugate to the fluctuating parameter. Analogous results for systems with continuous symmetry extend to $d \le 4$. In particular, for random-field models we rigorously prove uniqueness of the Gibbs state in 2D Ising systems, and absence of continuous symmetry breaking in the Heisenberg model in $d \le 4$, as predicted by Imry and Ma. Another manifestation of the phenomenon is found in 2D random-bond Potts models where a phase transition persists, but ceases to be first order.

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We report here on general results[1,2] concerning the thermodynamics of systems with quenched disorder. The main focus in this Letter is on a rigorous proof that in $d \le 2$ dimensions the presence of random fluctuations in the structural parameters results in the suppression of first-order phase transitions, i.e., elimination of discontinuities in the thermodynamic expectation values of the conjugate quantities. Discontinuities related to continuous symmetry breaking, as in the Heisenberg model at vanishing external field, are eliminated by arbitrarily weak quenched disorder even in the higher dimensions: $d \le 4$. Our analysis of these phenomena is tied in with some general results on the fluctuations of extensive functions of random couplings (such as the free energy) which apply in all dimensions. Some of the further applications of the method are mentioned at the end of the Letter.

A particular manifestation of the rounding of the first-order transitions in 2D is seen in the ferromagnetic random-field Ising spin models (RFIM), for which the persistence of a phase transition (at weak enough disorder) in $d > 2$ dimensions has now been rigorously established. The early prediction of this effect was given by Imry and Ma,[3] on the basis of the heuristic argument that the state of such a system is determined by the competition of the random field, whose cumulative effect on a uniform spin configuration in a region $\Lambda = [-L,L]^d$ is of the order $|\Lambda|^{1/2} = L^{d/2}$, with the symmetry-breaking mechanism, whose strength is of the order of the boundary $|\partial\Lambda| = L^{d-1}$. For the marginal case of $d=2$ dimensions, the prediction has been that the fluctuating field will have the dominant effect. If the distinct phases are related—in the absence of the quenched disorder—by a continuous symmetry, soft modes reduce the effect of the boundary conditions to L^{d-2}, and hence the marginal dimension is $d=4$.

The insufficiency of the above appealing argument became evident when a need arose to decide between its prediction and that of a "dimensional-reduction" principle, which suggests a higher value for the lower critical dimension of the RFIM. The resolution of the impasse required a series of rigorous works—starting with Ref. 4, and culminating with Imbrie[5] and Bricmont and Kupiainen[6]—which prove the stability (under weak disorder) of the Peierls mechanism (of symmetry breaking in the ground state,[5] and at low temperatures[6]) in $d > 2$ dimensions. The methods employed there could not, however, establish the disappearance of the symmetry breaking in 2D, which the results reported here prove for arbitrarily weak random fields. Though that part of the Imry-Ma prediction was not challenged by the alternative argument, it has been argued that it also deserves rigorous study—in particular since it concerns a qualitatively distinct *nonperturbative* effect.

Another example of the rounding of a first-order phase transition is the suppression of the discontinuity in the energy density[7] in random-bond Potts models. In this case, if the couplings remain ferromagnetic (or just "sufficiently" so[8]), the effect is just a change in the order of the transition, since at low temperatures the model exhibits symmetry breaking, and long-range order, even in the presence of randomness. The vanishing of the latent heat was also independently noted in a parallel work of Hui and Berker,[9] where it is derived—along with some further information on the nature of the phase transition—by renormalization-group (nonrigorous) arguments.

Our analysis is based on the consideration of fluctuations. The problem at hand may be presented as a question of the vanishing of an order parameter, M. The argument rests on the observation that the nonvanishing of M carries the implication that some finite-volume quantity, G_Λ, exhibits fluctuations which exceed an upper limit set by some other considerations. A contradiction is avoided only if $M=0$. The basic argument requires a certain refinement in the marginal dimensions. That strategy may be useful for a variety of problems.

To make the general notation transparent, let us start with some examples. In all cases, we consider systems of variables $\sigma = \{\sigma_x\}$ located on a d-dimensional lattice, say $\mathbb{Z}^d$, whose Hamiltonian is a sum of a translation-invariant, nonrandom, interaction and a fluctuating term with quenched randomness—represented here by a collection of independent random variables $\{\eta\}$ with a translation-invariant distribution.

(i) Random-field (RF) models,[10] with bounded spin variables ($|\sigma_x| \leq S < \infty$) and the Hamiltonian

$$H(\sigma) = -\tfrac{1}{2}\sum_{x,y} J_{x-y}\sigma_x\cdot\sigma_y - \sum_x (h+\epsilon\eta_x)\cdot\sigma_x . \quad (1)$$

In the ferromagnetic RFIM, σ_x are Ising spins and $J_z \geq 0$ (that condition is not relevant for the main result, though it does play a role in the translation of the lack of discontinuity in the magnetization to a statement on the uniqueness of the Gibbs state). In O(N) models ($N=3$ for the Heisenberg model), σ_x are N-component unit vectors, with the rotation-invariant *a priori* distribution. In that case the center dot in (1) represents a scalar product.

In the RF models the spins are subjected to a fluctuating magnetic field which is presented as a sum of two terms: one uniform (h), and the other random, with the order of magnitude of ϵ. The random fields (N-dimensional vectors) are independently distributed, with a probability measure $\nu(d\eta)$ about which we assume [with average denoted by $\mathcal{A}(f) \equiv \int f\nu(d\eta)$]:

$$\mathcal{A}(\eta)=0,\quad \mathcal{A}(\eta^2)>0 ,$$

and (2)

$$\mathcal{A}(e^{s\eta}) < \infty \ \text{for all}\ |s| < \infty .$$

E.g., η may be Gaussian; however, the reflection symmetry is relevant for us only in the discussion of the continuous symmetry breaking in $2 < d \leq 4$.

(ii) Random-bond (RB) q-state Potts models, with σ taking values in $\{1,\ldots,q\}$ and the Hamiltonian having the form of either

$$H_1(\sigma) = -\tfrac{1}{2}\sum_{x,y}(J_{x-y}+\epsilon_{x-y}\eta_{x,y})\delta_{\sigma_x,\sigma_y} , \quad (3)$$

or

$$H_2(\sigma) = -\tfrac{1}{2}\sum_{x,y}(1+\epsilon\eta_x+\epsilon\eta_y)J_{x-y}\delta_{\sigma_x,\sigma_y} = H_0(\sigma) - \sum_x \epsilon\eta_x\left[\sum_y J_{x-y}\delta_{\sigma_x,\sigma_y}\right] . \quad (4)$$

It is always assumed here that the distribution of the random parameters $\{\eta\}$ is translation invariant. In the case of H_1 that means that the η's form a number of classes—one for each value of $z=x-y$, with members of each class being identically distributed and also sharing a common strength parameter ϵ_z.

(iii) Spin-glass models, such as the Ising model with the Hamiltonian

$$H(\sigma) = -\tfrac{1}{2}\sum_{|x-y|=1}\eta_{x,y}\sigma_x\sigma_y - \sum_x (h+\epsilon\eta_x)\sigma_x , \quad (5)$$

which is of course not unrelated to (3). Here the inverse temperature (β) is akin to a second ϵ parameter.

In general, and unifying terms, the models consist of spins arranged on a lattice, with a Hamiltonian of the form

$$H(\sigma) = H_0(\sigma) + \sum_\alpha\sum_x (h_\alpha+\epsilon_\alpha\eta_{\alpha,x})\cdot g_\alpha(T_x\sigma) , \quad (6)$$

where the index α may parametrize pair-interaction terms of given range or other multiple-spin terms, g_α are bounded functions of the spin configuration, T_x are the translation operators (not to be confused with the temperature $T\equiv 1/\beta$), and $\eta_{\alpha,x}$ a collection of independent random variables, satisfying the conditions (2), with an identical distribution within each α class. The questions we address relate to the properties of the infinite-volume Gibbs states (of the variables σ) for typical configurations of the parameters $\{\eta_{\alpha,x}\}$.

Bulk properties of random systems are related to the free energy, F, which is derived from the finite-volume partition functions Z_Λ. By standard arguments, for almost every configuration of the parameters $\{\eta_{\alpha,x}\}$, the latter converges in the thermodynamic limit to a nonrandom function:[11]

$$\lim_{\substack{\Lambda=[-L,L]^d \\ L\to\infty}} \frac{T}{|\Lambda|}\ln Z_\Lambda(T,\{h\},\{\epsilon\},\{\eta\}) = F(T,\{h\},\{\epsilon\}) . \quad (7)$$

The free energy is convex in $\{h\}$, for fixed T and $\{\epsilon\}$, and hence it has directional derivatives. Any discontinuity of those corresponds to a first-order phase transition. One has, therefore, the following family of natural order parameters:

$$M_\alpha(T,\{h\},\{\epsilon\}) = \frac{1}{2}\left[\frac{\partial}{\partial(h_\alpha+0)} - \frac{\partial}{\partial(h_\alpha-0)}\right]F(T,\{h\},\{\epsilon\}) . \quad (8)$$

The definition of M has a natural extension to zero temperature, $T=0$, with $F(T=0,\ldots)$ interpreted as the ground-state energy density.

In the case of the ferromagnetic RFIM, model (1),

$$M(T,h,\epsilon) = \tfrac{1}{2}[\mathcal{A}(\langle\sigma_0\rangle_{\eta,+}) - \mathcal{A}(\langle\sigma_0\rangle_{\eta,-})] , \quad (9)$$

where $\langle\cdots\rangle_{\{\eta\},\pm}$ are the two well known extremal Gibbs states ("pure phases") constructed via standard choices ("+" or "−") of boundary conditions. These two states bracket all other Gibbs states (in the sense of Fortuin, Kasteleyn, and Ginibre[12]), with the implication that if $M(T,h,\epsilon)=0$ then, at those values of (T,h,ϵ), the Gibbs state is unique for almost every realization of $\{\eta_x\}$.

The following is our main result for the general case:[2]

Theorem 1.—In a ($d\leq 2$)-dimensional system with quenched disorder, with a Hamiltonian (6) satisfying the decay condition (12) (stated below), and with a continuous (i.e., "nonatomic") probability measure $\nu(d\eta)$,

$$M_a(T,\{h\},\{\epsilon\})=0 \quad \text{for all } T\geq 0,\ \{h\},\ \{\epsilon\}, \text{ and } \alpha \text{ for which } \epsilon_a>0. \tag{10}$$

In the Ising and Potts ferromagnetic RF and RB models, (10) holds at $T>0$ regardless of the continuity assumption on the random-field and coupling distribution $\nu(d\eta)$.

For the ferromagnetic RFIM a rather direct implication (by the Fortuin-Kasteleyn-Ginibre[12] domination arguments) is that in two dimensions at any fixed magnetic field h (in particular $h=0$), and temperature $T>0$, for almost every random-field configuration $\{\epsilon\eta_x\}$ (with $\epsilon>0$) the system has a unique Gibbs state. If the probability distribution $\nu(d\eta)$ is continuous then H has also a unique infinite-volume ground-state configuration. Furthermore, when h and T are varied with $\{\epsilon\eta_x\}$ fixed, then almost surely the system has no bulk first-order phase transitions at $T\geq 0$, with the case $T=0$ subject to the above restriction. (A bulk first-order phase transition occurs when a system has distinct Gibbs states which differ on the translation-averaged expectation value of some local quantity. See Ref. 2 for further discussion.)

Thus, the situation for $\epsilon>0$ is drastically different from the case $\epsilon=0$. The restriction to continuous $\nu(d\eta)$ is not totally superfluous at $T=0$ (though it could be eased), since in the RF model if $\nu(d\eta)$ has a discrete component then at least for certain values of $\epsilon>0$ the ground state is typically nonunique—with a macroscopic degeneracy. On the other hand, for $T>0$ we expect that restriction to be unnecessary even in the general case.

The models to which Theorems 1 and 2 (below) apply include systems with long-range interactions, limited by the conditions suggested by the heuristic Imry-Ma arguments. For pair interactions, with J_{x-y} falling off by a power law, we require (in both theorems)

$$|J_{x-y}|\leq \text{const}/|x-y|^{3d/2}. \tag{11}$$

In stating the general assumptions for Theorem 1 we use the following notation. For a function $\psi(\sigma)$, and A a subset of the lattice, we denote the amplitude of the dependence of $\psi(\cdot)$ on the spins in A by $\mathcal{O}_A\psi \equiv \sup\{\psi(\sigma)-\psi(\sigma')\,|\,\sigma_{A^c}=\sigma'_{A^c}\}$. Our condition on the Hamiltonian (6) is that for rectangular regions Λ

$$\mathcal{O}_{\Lambda^c}H_{0;\Lambda},\ \sum_{\alpha;x\in\Lambda}\mathcal{O}_{T_x\Lambda^c}g_\alpha\leq \text{const}\times|\Lambda|^{1/2}, \tag{12a}$$

where $H_{0;\Lambda}$ is that part of H_0 which involves spins in Λ. E.g., in the often used representation[13]

$$H_0=\sum_{A\subset\mathbb{Z}^d}\psi_A(\sigma_A)$$

the first of the two conditions in (12a) is

$$\sum_{A\in 0,\,\text{diam}(A)\geq L}\frac{\text{diam}(A)}{|A|}\sup_\sigma|\psi_A(\sigma)|\leq \text{const}\times L^{d/2}/L^{d-1}=\text{const}\times L^{1-d/2}; \tag{12b}$$

and a similar statement can be made for g.

The general statement derived in Ref. 2 on the inhibiting effects of quenched disorder on the *continuous* symmetry breaking includes the following:

Theorem 2.—In the O(N) random-field models, with measures $\nu(d\eta)$ which are symmetric under the reflection $\eta\to-\eta$, with continuous projections on each line [i.e., for each fixed N-dimensional vector M and each real r, $\nu(\{\eta\cdot M=r\})=0$]

$$M(T,h=0,\epsilon)=0 \quad \text{for all } T\geq 0,\ \epsilon>0, \tag{13}$$

provided J_{x-y} satisfies (11) and the dimension is $d\leq 4$.

We shall not give here the full details of the proof of Theorem 1, but instead outline the main steps in the context of the ferromagnetic random-field Ising model. For that model our $G_\Lambda(T,h,\{\epsilon\eta\})$ is very similar to the difference in the free energy between the + and the − boundary conditions:

$$\mathbb{G}_\Lambda=T\ln Z_{\Lambda,+}(T,h,\{\epsilon\eta\})-T\ln Z_{\Lambda,-}(T,h,\{\epsilon\eta\}).$$

We choose to consider a quantity constructed in a seemingly more complicated way, since that construction yields some very convenient translation-invariance properties. The definition of G_Λ for the general case is somewhat more involved;[2] however, even for the general case it is possible to construct G_Λ satisfying all the properties which are essential for the argument described below. For the RFIM we define

$$G_\Lambda(T,\{h\},\{\epsilon\eta\})=\frac{T}{2}\ln\left\langle\exp\left[\beta\sum_{x\in\Lambda}\epsilon\eta_x\sigma_x\right]\right\rangle_{\eta,-}-\frac{T}{2}\ln\left\langle\exp\left[\beta\sum_{x\in\Lambda}\epsilon\eta_x\sigma_x\right]\right\rangle_{\eta,+}, \tag{14}$$

where one should note that the terms in the exponents cancel out similar terms present in the Gibbs factors of the states $\langle\cdots\rangle_{\eta,\pm}$. By that observation, for each $x\in\Lambda$

$$\epsilon^{-1}\partial G_\Lambda/\partial\eta_x=\tfrac{1}{2}[\langle\sigma_x\rangle_{\eta,+}-\langle\sigma_x\rangle_{\eta,-}], \tag{15}$$

and hence, using (9),

$$\mathcal{A}(\partial G_\Lambda/\partial\eta_x)=\epsilon M(T,h,\{\epsilon\eta\}). \tag{16}$$

We shall now describe two conflicting bounds on the magnitude of fluctuations of G_Λ. The first observation is that for short-range interactions, G_Λ has the order of magnitude of the boundary,

$$G_\Lambda(T,h,\{\epsilon\eta\}) \le A\,|\partial\Lambda| \,, \tag{17}$$

with some finite constant A.

The "*a priori*" bound (17) will be contrasted with an implication of (16)—which may be taken to suggest that if the order parameter M does not vanish then the quantity G_Λ has significant fluctuations, with variance of the order of the volume $|\Lambda|$. More precisely, the combination of (16) with general variance bounds presented in Refs. 1 and 2, leads to

$$\lim_{\substack{\Lambda + [-L,L]^d \\ L\to\infty}} \mathcal{A}(G_\Lambda^2/|\Lambda|) \ge \epsilon^2\theta_\nu^2(M)\,. \tag{18}$$

with the ν-dependent function[2]

$$\theta_\nu(m) = \inf\left\{ \left[\int \nu(d\eta) g(\eta)^2\right]^{1/2} \middle|\, g\in C^1, |g'(\cdot)| \le 1, \int \nu(d\eta) g'(\eta) = m \right\}. \tag{19}$$

The most relevant property of θ is that, for any continuous probability measure ν, $\theta_\nu(m)$ is strictly positive at $m\neq 0$. (A separate argument is required for the discrete case.)

In $d=2$ dimensions the bounds (17) and (18) are consistent only if either $M=0$, or if the quantity G_Λ manages to attain (with non-negligible probability) the order of magnitude $O(|\Lambda|^{1/2})$ without ever exceeding a certain multiple of $|\Lambda|^{1/2}$. One may expect that feat to be rather difficult for an extensive quantity, since its distribution may share some qualitative features with the Gaussian. In fact, we have also derived the following significantly stronger bound (valid in any dimension), whose relation to (18) is in the spirit of the last comment:

$$\lim_{\substack{\Lambda = [-L,L]^d \\ L\to\infty}} \mathcal{A}(\exp(tG_\Lambda/|\Lambda|^{1/2})) \ge \exp[\tfrac{1}{2}t^2\epsilon^2\theta_\nu^2(M)] \text{ for all } t\ge 0\,. \tag{20}$$

That bound is plainly inconsistent with (17), unless $M=0$, since (17) implies (for $d=2$, and regular Λ)

$$\mathcal{A}(\exp(tG_\Lambda/|\Lambda|^{1/2})) \le \exp(tA), \text{ for all } t\ge 0\,. \tag{21}$$

Hence Theorem 1, for the finite-range RFIM. For long-range interactions, the bound (17) may not hold; however, condition (12) assures (for $d=1,2$) that G_Λ is still of the order $O(|\Lambda|^{1/2})$, which is what matters here.

Our derivation of Theorem 2, concerning systems with a continuous symmetry, is based on a combination of the above method with a Herring-Kittel-type[14] argument, as elucidated by Pfister[15] in his rigorous proof of the classical Mermin-Wagner[16] phenomenon.

References 1 and 2 also contain some useful general results on the nature of the fluctuations of extensive quantities, and further applications of the approach based on their study. The applications include a proof (restricted to $d=2$ dimensions) that all the ground states of nondegenerate 2D spin-glass models [with continuous $\nu(\cdot)$] are regionally congruent, in the Fisher-Huse[17] sense.[2]

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