Walking Technicolor beyond the Ladder Approximation

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(Received 13 February 1989)

We prove that in the scale-invariant limit of walking technicolor the fermion dynamical mass function falls off exactly as $1/p$ when the gauge coupling α just reaches the critical value required to trigger spontaneous chiral-symmetry breaking. The proof is given to all orders in α.

PACS numbers: 12.50.Lr, 11.30.Qc, 11.30.Rd

An attractive resolution of the flavor-changing neutral-current problem in extended technicolor[1,2] is provided by the walking-technicolor scenario.[3-5] In the ladder approximation to walking technicolor, it has been shown that the fermion dynamical mass falls off much more slowly with increasing momentum than in precociously asymptotically free theories. The slowly falling dynamical mass enhances the technifermion condensate. In extended technicolor theories the ordinary-fermion masses are proportional to the technifermion condensate. Its enhancement therefore allows the extended technicolor scale to be raised to adequately suppress the flavor-changing neutral-current amplitudes and yet generate reasonable masses for the ordinary fermions.

The crucial ingredient in the resolution of the flavor-changing neutral-current problem in walking technicolor is the $1/p$ behavior of the dynamical mass over a large range of momentum. Yet this behavior of the dynamical mass was obtained only in ladder approximation when the gauge coupling α was set equal to the lowest-order value $\pi/3C_2(R)$ for the critical coupling α_c. It is this feature of the ladder analysis that raises some serious concern about its validity beyond the lowest order.[6] Is the coincidence of the $1/p$ behavior of the dynamical mass with $\alpha=\alpha_c$ an artifact of the lowest-order approximation or does it persist to all orders? This question will be addressed in this paper. We shall prove that, to all orders in perturbation theory, the fermion dynamical mass $\Sigma(p)/A(p)$ in the technifermion propagator $S(p)=i/[\not{p}A(p)-\Sigma(p)]$ falls off exactly as $1/p$ when the gauge coupling α just reaches the critical value required to trigger spontaneous chiral-symmetry breaking. The proof will be based upon the stability analysis[7] of the Cornwall-Jackiw-Tomboulis (CJT) effective potential $\Gamma[S]$.[8] The reason behind this choice of the effective potential is that the extremum condition $\delta\Gamma[S]/\delta[\Sigma(p)/A(p)]=0$ gives the gap equation for $\Sigma(p)/A(p)$, and this result will turn out to be very useful in our proof.

We shall consider the scale-invariant (static) limit of a walking-technicolor scenario. In a realistic walking-technicolor scenario the gauge coupling $\alpha(p)$ runs very slowly between the chiral-symmetry-breaking scale μ and some very large momentum Λ $(\Lambda\gg\mu)$. The gauge coupling, however, resumes its normal running beyond Λ. By scale-invariant limit of walking technicolor we mean the following: We shall let Λ approach infinity and also assume that the gauge coupling $\alpha(p)$ does not run all the way up to infinity. This artificiality will be invoked only to simplify our analysis. In a realistic walking-technicolor scenario we cannot ignore the running of the gauge coupling $\alpha(p)$ for momenta $p>\Lambda$. So strictly speaking our results should be regarded as valid only for momenta p which satisfy the constraint $\mu<p<\Lambda$. Nonlinear effects will become important for momenta $p<\mu$ and modify the results of our linearized analysis and the running of the gauge coupling above Λ will give the dynamical mass a much softer asymptote. The choice of the gauge parameter deserves some comment. For a non-Abelian gauge theory a static gauge coupling $[\beta(\alpha)=0]$ does not ensure scale invariance unless the gauge parameter a is also static $[\beta_g(\alpha,a)=0]$. To make $\beta_g(\alpha,a)=0$ to all orders in perturbation theory we shall work in Landau gauge $(a=0)$.

The CJT effective potential $\Gamma[S]$ is given by

$$\Gamma[S]=-i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}[\ln S_0^{-1}(p)S(p)-S_0^{-1}(p)S(p)+1]+\Gamma_2[S]\,, \tag{1}$$

where $S_0(p)=i/\not{p}$ and $\Gamma_2[S]$ is the sum of all two-particle-irreducible (2PI) vacuum graphs. The first term on the right-hand side of Eq. (1) will be referred to as the kinetic term and the second term ($\Gamma_2[S]$) as the interaction term.

For the stability analysis about the symmetric extremum we need to consider only the quadratic term in the expansion of $\Gamma[S]$ in powers of $\Sigma(p)/A(p)$. Although this truncation of $\Gamma[S]$ may not be very accurate below the chiral-symmetry-breaking scale where nonlinear effects become important, this approximation should be sufficient to investigate the onset of chiral-symmetry breaking starting from the linear regime $[\Sigma(p)\ll p]$. In this linearized approximation $\Gamma_2[S]$ receives contribution from two different kinds of diagrams. In the first kind two factors of $\Sigma(p)/A(p)$ are inserted on one and the same fermion line, leaving the remaining fermion lines with zero mass. The massless part of this diagram contains the sum of all self-energy diagrams of the technifermion and can be shown to be equal to the wave-function renormalization $i[A(p)-1]\not p$. The contribution of the first diagram to $\Gamma_2[S]$ is then given by

$$4nd\int\frac{d^4p}{(2\pi)^4}\frac{[A(p)-1]}{A(p)p^2}\frac{\Sigma^2(p)}{A^2(p)},$$

where n is the number of flavors and d is the dimension of the representation of technifermions. This contribution will be added to the kinetic part of $\Gamma[S]$. The second kind of diagram has the two factors $\Sigma(p)/A(p)$ and $\Sigma(k)/A(k)$ inserted on different fermion lines, leaving the remaining fermion lines with zero mass. The massless part of the latter diagram can be identified with the complete 2PI fermion-antifermion scattering kernel $K(p,k;\alpha)$. The anomalous scaling of the kernel $K(p,k;\alpha)$ can be factored out using renormalization-group methods and evaluated in the massless limit.[9] In a scale-invariant theory the kernel $K(p,k;\alpha)$ then takes the following general form

$$K(p,k;\alpha)=\frac{1}{M^2}A^2(M)F\left(\frac{m^2}{M^2}\right), \tag{2}$$

where $M=\max(p,k)$ and $m=\min(p,k)$. With the help of these ingredients it is straightforward to show that to all orders in perturbation theory the quadratic term in the expansion of $\Gamma[S]$ about $\Sigma(p)=0$ is given by

$$\Gamma[S]=\frac{nd}{4\pi^2}\left(\int_0^\infty p\frac{\Sigma^2(p)}{A^2(p)}dp-\int_0^\infty p\frac{\Sigma(p)}{A(p)}dp\int_0^\infty\frac{k}{M^2}\frac{A(M)}{A(m)}F\left(\frac{m^2}{M^2}\right)\frac{\Sigma(k)}{A(k)}dk\right). \tag{3}$$

To exploit the scale invariance of the theory we follow Peskin and define new variables as follows

$$\frac{p}{\mu}=e^{\xi},\quad \frac{k}{\mu}=e^{\eta},\quad \frac{\Sigma(p)}{A(p)}=\frac{1}{p}\sigma(\xi), \tag{4}$$

where μ is some reference momentum which for the sake of definiteness will be chosen to be equal to the chiral-symmetry-breaking scale. In terms of ξ and η we get

$$\Gamma[S]=\frac{nd}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sigma(\xi)[\delta(\xi-\eta)-G(\xi,\eta)]\sigma(\eta)d\xi\, d\eta, \tag{5}$$

where

$$G(\xi,\eta)=e^{-[1+\gamma_2(\alpha)]|\xi-\eta|}F(e^{-2|\xi-\eta|})$$

and $\gamma_2(\alpha)$ is the anomalous dimension of the technifermion field. $\delta(\xi-\eta)$ represents the contributions of the kinetic term and the interaction term of first kind. $G(\xi,\eta)$ represents the contribution of the interaction term of second kind. Note that the scale invariance in the variables p,k is reflected as translational invariance in the variables ξ,η and this requires $G(\xi,\eta)$ to be a function of $\xi-\eta$ only. The additional exchange symmetry $(p\leftrightarrow k)$ of the kernel $K(p,k;\alpha)$ makes $G(\xi,\eta)$ a function of $|\xi-\eta|$ only.

Now $\psi_q(\xi)=e^{iq\xi}/(2\pi)^{1/2}$ is an eigenfunction of the kernel $[\delta(\xi-\eta)-G(|\xi-\eta|)]$ with the eigenvalue

$$\lambda(q)=1-2\int_{-\infty}^{+\infty}e^{-2iqy}G(2|y|)dy.$$

The eigenvalues $\lambda(q)$ are real and they form a continuous spectrum which is degenerate $[\lambda(q)=\lambda(-q)]$. The eigenfunctions $\psi_q(\xi)$ form a complete set of orthonormal functions. Therefore an arbitrary variation $\sigma(\xi)$ or $\Sigma(p)/A(p)$ about the origin can be expanded in terms of $\psi_q(\xi)$ as

$$\sigma(\xi)=\int_{-\infty}^{+\infty}\hat\sigma(q)\psi_q(\xi)dq,\quad \frac{\Sigma(p)}{A(p)}=\int_{-\infty}^{+\infty}\frac{1}{p}\hat\sigma(q)\psi_q\left(\ln\frac{p}{\mu}\right)dq, \tag{6}$$

and the quadratic form in Eq. (5) is then diagonalized to

$$\int\sigma(\xi)[\delta(\xi-\eta)-G(|\xi-\eta|)]\sigma(\eta)d\xi\, d\eta=\int\hat\sigma(q)\lambda(q)\hat\sigma(-q)dq. \tag{7}$$

In Landau gauge to lowest order in perturbation theory $G(2|y|)\geq 0$ for $-\infty\leq y\leq+\infty$. Analytical and numerical studies show that this is true for the $O(\alpha^2)$ kernel also, when α is close to α_c.[10] In this paper we shall assume that this feature of the kernel remains true to all orders in perturbation theory. Then it follows that for $q\neq 0$,

$$\lambda_0(q)=\int_{-\infty}^{+\infty}e^{-2iqy}G(2|y|)dy<\int_{-\infty}^{+\infty}G(2|y|)dy=\lambda_0(0), \tag{8}$$

and $\lambda(q)=1-2\lambda_0(q)>\lambda(0)>0$. Recall that $G(2|y|)$ represents the contribution of the interaction term. Therefore if we start from the weak-coupling phase and increase α, the eigenvalue that first reaches zero is $\lambda(0)$. The critical coupling is therefore determined by setting $\lambda(0)=0$ and the eigenvalue equation for the zero mode then becomes

$$\int_{-\infty}^{+\infty} G(|\xi-\eta|)d\eta=\int_0^\infty \tilde{G}(p,k)\frac{dk}{k}=1\,. \tag{9}$$

To prove that Eq. (9) implies that $\Sigma(p)/A(p)$ behaves as $1/p$ at criticality, we note that the gap equation is obtained by setting $\delta\Gamma[\sigma]/\delta\sigma(\xi)=0$. In terms of the original variables p and k this gives us

$$p\frac{\Sigma(p)}{A(p)}=\int_0^\infty \tilde{G}(p,k)k\frac{\Sigma(k)}{A(k)}\frac{dk}{k}\,. \tag{10}$$

$\tilde{G}(p,k)$ is a positive, dimensionless, symmetric function of p and k. This implies that $\tilde{G}(p,k)=\tilde{G}(x)$ where $x=m/M$. If we assume that at criticality $\Sigma(p)/A(p)\propto 1/p^{1+\nu}$, then consistency of Eqs. (9) and (10) at criticality requires that

$$\int_0^1 \tilde{G}(x)(x^\nu+x^{-\nu}-2)\frac{dx}{x}=0\,. \tag{11}$$

Consider the function $f(x)=x^\nu+x^{-\nu}-2$ for $\nu\geq 0$. It is sufficient to consider this case only because $f(x)$ is symmetric under $\nu\to-\nu$. We find that $f(0)=\infty$, $f(1)=0$, and $f'(x)\leq 0$ for $0\leq x\leq 1$. This implies that $f(x)\geq 0$ for $0\leq x\leq 1$ and Eq. (11) can then be satisfied if and only if $f(x)=0$ for all x between 0 and 1, i.e., $\nu=0$ or $\Sigma(p)/A(p)\propto 1/p$.

It is important to examine if our condition for criticality is gauge invariant, in spite of the restriction of working in Landau gauge. The question is: Is α_c defined by the equation $q(\alpha)=0$ gauge invariant? Suppose that for some $\alpha\geq\alpha_c$ the eigenvalues corresponding to the modes $q(\alpha)$ and $-q(\alpha)$ vanish. The eigenvalue equations of the corresponding modes then become

$$p\frac{1}{p}e^{\pm iq(\alpha)\ln(p/\mu)}=\int_0^\infty \tilde{G}(p,k)k\frac{1}{k}e^{\pm iq(\alpha)\ln(k/\mu)}\frac{dk}{k}\,. \tag{12}$$

Comparing Eqs. (10) and (12) we find that

$$\frac{\Sigma(p)}{A(p)}=\mu\left(\frac{\mu}{p}\right)e^{\pm iq(\alpha)\ln(p/\mu)}$$

satisfies the linearized gap equation. On the other hand, the operator-product expansion of the technifermion propagator[9,11] tells us that in the static limit $\Sigma(p)/A(p)=\mu(\mu/p)^{2-\gamma_m(\alpha)}$, where $\gamma_m(\alpha)$ is the anomalous dimension associated with the technifermion bilinear $\bar{\psi}\psi$. Comparing the last two equations we get $q(\alpha)=\pm i[1-\gamma_m(\alpha)]$. At criticality $q(\alpha)=0$ and this implies that $\gamma_m(\alpha)=1$ which is a gauge-invariant result order by order in perturbation theory if all renormalizations are performed in the minimal-subtraction scheme.[12] Note that in the strong coupling phase $(\alpha>\alpha_c)$ $\gamma_m(\alpha)$ is complex and has the form $1\pm i\chi(\alpha)$.[13] This makes $q(\alpha)=\pm i[1-\gamma_m(\alpha)]$ real. We can also take appropriate linear combination of the eigenfunctions $(1/p)\exp\{i[q(\alpha)\ln(p/\mu)+\delta]\}$ and $(1/p)\times\exp\{-i[q(\alpha)\times\ln(p/\mu)+\delta]\}$ and obtain the oscillatory solution

$$\frac{\Sigma(p)}{A(p)}\propto\frac{1}{p}\sin\left[q(\alpha)\ln\frac{p}{\mu}+\delta\right]$$

for $\alpha>\alpha_c$. The phase δ is to be determined by requiring that the oscillatory solution in the momentum regime $\mu<p<\Lambda$ matches smoothly with the asymptotic behavior. The value of δ will therefore be different for mechanical mass and dynamical mass.

In conclusion, we have proved that in the scale-invariant limit of walking technicolor, to all orders in perturbation theory, the anomalous dimension $\gamma_m(\alpha)$ becomes equal to unity when the gauge coupling α reaches the critical value α_c. The condensate enhancement, which is crucial for the suppression of flavor-changing neutral-current amplitudes in walking technicolor, is therefore not an artifact of ladder approximation but is true to all orders. On the other hand, explicit calculation of the effects of higher-order corrections in walking technicolor shows that these corrections are small.[10] This means that we can use the expressions for $\gamma_m(\alpha)$ and α_c in ladder approximation for phenomenological studies of walking technicolor and obtain quite reliable results. Finally the method outlined in this paper can be used to prove that in QED in three dimensions the dynamical mass $\Sigma(p)/A(p)$ falls off exactly as $1/p^{1/2}$ to all orders in $1/N$ expansion, when the number of fermions just reaches the critical value N_c that is required to trigger chiral-symmetry breaking.[14]

I thank Professor Kenneth Lane for suggesting the proof of the gauge invariance of the criticality condition, for careful reading of the manuscript, and for many other helpful discussions. The support and hospitality of the Physics Department of Boston University where this work was done is gratefully acknowledged. This work was supported in part by the Department of Energy under Contracts No. DE-AC02-76ER01545 and No. DE-AC02-86ER40284.

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