# Arnol'd Diffusion in Two Dimensions

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We consider the relativistic interaction of a charged particle with obliquely propagating waves of arbitrary polarization. We show that an arbitrarily small wave packet composed of waves with parallel phase velocity comparable to the speed of light (e.g., slow extraordinary mode) can, under certain conditions, accelerate particles to unlimited energy through a process of Arnol'd diffusion in two dimensions.

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The problem of wave-particle interaction in a strong magnetic field, $B_0$, is of fundamental importance in plasma physics and has been studied by various authors.[1-6] Recently, Zaslavskii *et al.*[7] studied the nonrelativistic interaction of a particle with an electrostatic wave packet propagating perpendicularly to the magnetic field. Such a system has $1\frac{1}{2}$ degrees of freedom. They showed that under certain conditions such an interaction results in a web structure in phase space through which the particle may be accelerated to large energies, by a process analogous to Arnol'd diffusion[8] in systems with three or more degrees of freedom. Soon after, however, Longcope and Sudan[9] showed through a relativistic treatment of the above study that particles can only be accelerated up to a critical energy by a wave packet above a critical amplitude. Strictly speaking, the relativistic mass dependence on energy leads to the appearance of Kolmogorov-Arnol'd-Moser (KAM) surfaces and the web only exists up to a finite energy. Here, we consider the interaction of a charged particle with an obliquely propagating wave packet of arbitrary polarization. We show that the correct relativistic limit of the problem studied by Zaslavskii *et al.*[7] occurs at angles $\alpha=\cos^{-1}(1/n)$, where $n$ is the refractive index of the plasma, and not at $\alpha=90°$. We find that Arnol'd diffusion to infinite energies is again possible for waves of infinitesimal amplitude. Furthermore, we show that the autoresonance acceleration mechanism[10,11] for a parallel propagating wave with $n=1$ is a special case of our general treatment.

The relativistic Hamiltonian of a charged particle of mass $m$ and charge $q$ in the presence of an obliquely propagating wave packet is

$$H=[(c\mathbf{P}-q\mathbf{A})^2+m^2c^4]^{1/2}+q\Phi\,, \tag{1a}$$

where the scalar and vector potentials of the $i$th wave are

$$\Phi_i=\Phi_{0i}\sin\psi_i\,,$$
$$\mathbf{A}_i=(A_{1i}\cos\alpha_i\sin\psi_i, A_{2i}\cos\psi_i\,, -A_{1i}\sin\alpha_i\sin\psi_i)\,, \tag{1b}$$

and

$$\Phi=\sum_{i=1}^{N}\Phi_i,\quad \mathbf{A}=\sum_{i=1}^{N}\mathbf{A}_i+xB_0\hat{\mathbf{e}}_y\,, \tag{1c}$$

with $k_{yi}=0$, $k_{\perp i}=k_i\sin\alpha_i$, $k_{\parallel i}=k_i\cos\alpha_i$, $\psi_i=k_{\perp i}x+k_{\parallel i}z-\omega_i t$, and $N$ the number of waves in the wave packet. We allow $A_{2i}$ to be positive or negative depending on whether the wave is right-hand or left-hand polarized, respectively. The system described in (1a) has, in general, more than two degrees of freedom, but reduces to two if $n_i\cos\alpha_i$ is the same for all the waves. Then we can eliminate the time dependence in $H$ through the generating function $F_2=[z-t(\omega_i/k_{\parallel i})]P_z'$. In order to facilitate the analysis further, we make a series of canonical transformations: $P_y$ is a constant of motion and is transformed away through $F_2=(x-cP_y/qB_0)P_x'$. Then, $P_y$ appears only in $\psi_i=k_{\perp i}x'+k_{\parallel i}z'+\delta_i$, where $\delta_i=k_{\perp i}cP_y/qB_0$ is a constant for each wave. Next, we expand the new Hamiltonian to first order in wave amplitudes. The final step is a transformation of $(x,P_x)$ into the zero-order action-angle variables $(\theta,J)$, where $P_x=(2|q|JB_0/c)^{1/2}\cos\theta$ and $x=(2Jc/|q|B_0)^{1/2}\sin\theta$. It is convenient to use the perpendicular momentum $P_\perp$ and the gyroradius $\rho$ instead of the action $J$, depending on the context. These are related to $J$ through $P_\perp=(2|q|JB_0/c)^{1/2}$ and $\rho=(2Jc/|q|B_0)^{1/2}$. Using standard Bessel function identities[12] to expand products like $\sin\theta\cos\psi_i$ and dropping the primes from the transformed variables, we have

$$H=H_0(P_\perp,P_z)+\sum_{i=1}^{N}\sum_{l=-\infty}^{l=+\infty}Z_{i,l}\sin(l\theta+k_{\parallel i}z+\delta_i)\,, \tag{2a}$$

where

$$H_0=mc^2\left[\gamma_0-\frac{P_z}{mc}\frac{1}{n_i\cos\alpha_i}\right], \tag{2b}$$

$$\gamma_0=\left[1+\frac{P_\perp^2+P_z^2}{m^2c^2}\right]^{1/2}, \tag{2c}$$

$$Z_{i,l}=mc^2\left\{\left(\frac{q\Phi_{0i}}{mc^2}-\frac{l}{k_{\perp i}\rho}\frac{qA_{1i}}{mc^2}\frac{P_\perp}{mc\gamma_0}\cos\alpha_i+\frac{qA_{1i}}{mc^2}\frac{P_z}{mc\gamma_0}\sin\alpha_i\right)J_l(k_{\perp i}\rho)+\left(\frac{|q|A_{2i}}{mc^2}\frac{P_\perp}{mc\gamma_0}\right)J_l'(k_{\perp i}\rho)\right\}. \tag{2d}$$

For simplicity, we consider particles with $P_y=0$. We first examine the one-wave case. We let $i\rightarrow 1$, and we drop the sum over $i$ in (2a). Suppose there is a resonance at a specific value of $l=l_1$. We transform to a rotating frame through the generating function $F_2=[z+(l_1/k_{\parallel 1})\theta]\hat{P}_z+\theta\hat{J}$, where the new variable $k_{\parallel 1}\hat{z}=k_{\parallel 1}[z+(l_1/k_{\parallel 1})\theta]$ measures the slow rotation around that resonance. Averaging over the fast angle $\hat{\theta}=\theta$ gives

$$\bar{H}=H_0(P_\perp,P_z)+Z_{1,l_1}\sin(k_{\parallel 1}\hat{z}). \qquad (3)$$

The fixed points of $\bar{H}$ are solutions of $\partial\bar{H}/\partial\hat{P}_z=0$ and $\partial\bar{H}/\partial\hat{z}=0$. The resonance condition is given by

$$\gamma_0-\frac{P_z}{mc}(n_1\cos\alpha_1)-l_1\frac{\Omega}{\omega_1}=\pm\frac{k_{\parallel 1}\gamma_0}{\omega_1}\frac{\partial Z_{1,l_1}}{\partial\hat{P}_z}. \qquad (4)$$

Here $\Omega$ is the nonrelativistic gyrofrequency. Expansion of $\bar{H}$ around the elliptic fixed point $(k_{\parallel 1}\hat{z}_s,\hat{P}_{z,s})$ yields

$$\bar{H}=\bar{H}(\hat{P}_{\perp,s},\hat{P}_{z,s})+\tfrac{1}{2}G(\hat{P}_z-\hat{P}_{z,s})^2+\tfrac{1}{2}F(k_{\parallel 1}\hat{z}-k_{\parallel 1}\hat{z}_s)^2,$$

where $F=\mp Z_{1,l_1}$ and

$$G=\frac{1}{m\gamma_0}\left(1-\frac{1}{n_1^2\cos^2\alpha_1}\right)\pm\frac{\partial^2 Z_{1,l_1}}{\partial\hat{P}_z^2}=G^{(0)}+G^{(1)}. \qquad (5)$$

$F$ and $G$ are evaluated at the elliptic fixed point. The system is called intrinsically degenerate[13] if $|G^{(0)}|<|G^{(1)}|$ and accidentally degenerate if the inequality is reversed. The trapping width is given by $\Delta P_z=2\times|F/G|^{1/2}$ and the bounce frequency by $\omega_b=k_{\parallel 1}n\times|FG|^{1/2}$.

We see from (2b) and (4) that if $n_1\cos\alpha_1=1$, the surfaces of zero-order Hamiltonian and zero-order resonance are parabolic and either lie on top of each other (if $H_0/mc^2=l_1\Omega/\omega_1$), or do not intersect anywhere (if $H_0/mc^2\neq l_1\Omega/\omega_1$). Thus, there can be at most one resonance in this case. We note that the nonrelativistic intrinsic degeneracy condition, namely $\omega_1=l_1\Omega$ and $\alpha=90°$, has changed to $\omega_1=l_1\Omega(mc^2/H_0)$ and $\alpha_1=\cos^{-1}(1/n)$ in the relativistic case.

When $\omega_1=l_1\Omega mc^2/H_0$, we can plot contours of constant $\Delta\bar{H}=\bar{H}-H_0$ which unravel the phase-space structure (Fig. 1). Note, however, that any untrapped region below the lowest-lying first-order islands in phase space would not be described by surfaces of constant $\bar{H}$. The phase space is very similar to the nonrelativistic case studied by Fukuyama *et al.*[3] and others, with the notable exception that the relativistic first-order island widths are several orders of magnitude larger than the nonrelativistic ones. The similarity in the phase space is not too surprising since $\Delta\bar{H}$ has the same exact form in the two cases.

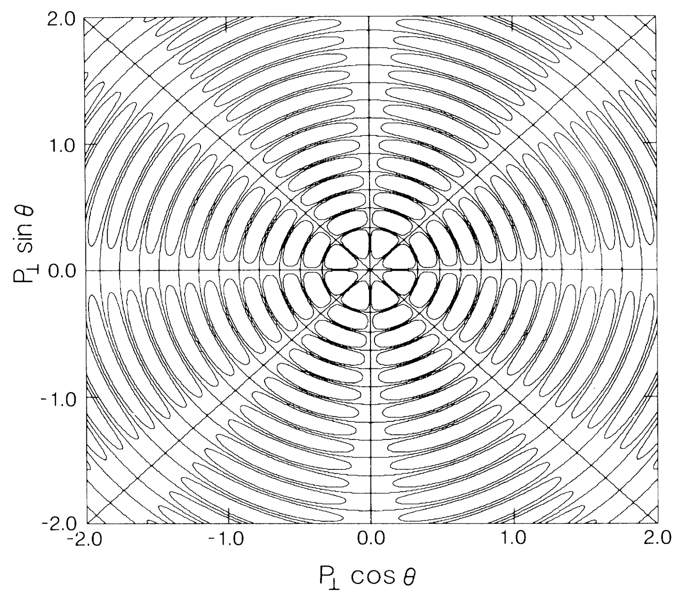


FIG. 1. Phase-space structure in the $P_x$-$P_y$ plane. One wave with $\omega_1=4\Omega$, $\alpha_1=80°$, $H_0=mc^2$, and $n_1\cos\alpha_1=1$. Notice the large first-order islands which can accelerate particles coherently to very large energies.

The autoresonance particle acceleration[10,11] by parallel propagating electromagnetic waves with $n=1$, which has recently[14,15] been discussed in relation to the Cyclotron Autoresonance Maser (CARM), is a special case of the relativistic intrinsically degenerate case discussed here. For $\alpha=0°$ and $n=1$, the motion is integrable and $G=0$ to all orders, resulting in an infinitely large trapping width.

We now resume our study of the wave-packet case. We consider a wave packet comprising of waves all satisfying the conditions $n_i\cos\alpha_i=1$ and $\omega_i=l_i\Omega mc^2/H_0$. This constitutes the relativistic analog of the system studied by Zaslavskii *et al.*[7] Following an averaging procedure as in the one-wave case, we are able to use the same generating function to get the averaged Hamiltonian:

$$\bar{H}=H_0(P_\perp,P_z)+\sum_{i=1}^{N}Z_{i,l_i}\sin(k_{\parallel i}\hat{z}). \qquad (6)$$

The position of the fixed points and formulas for the trapping width and the bounce frequency are obtained from a generalization of the expressions for the one-wave case. The number of fixed points, however, increases in general and the structure in phase space becomes more complicated.

Now we can show that the formation of web is limited below a critical energy in the case when $\alpha_i=90°$. The transition from accidental to intrinsic degeneracy for the case of an electrostatic wave packet occurs when

$$\frac{1}{m\gamma_0}|n_i^2\cos^2\alpha_i-1|\leq q\Phi_0\left(\frac{ck_{\perp i}}{m\Omega^2}\right)^2\frac{|\sum J_{l_i}''(k_{\perp i}\rho)\sin(k_{\parallel i}\hat{z})|}{\rho^2}. \qquad (7)$$

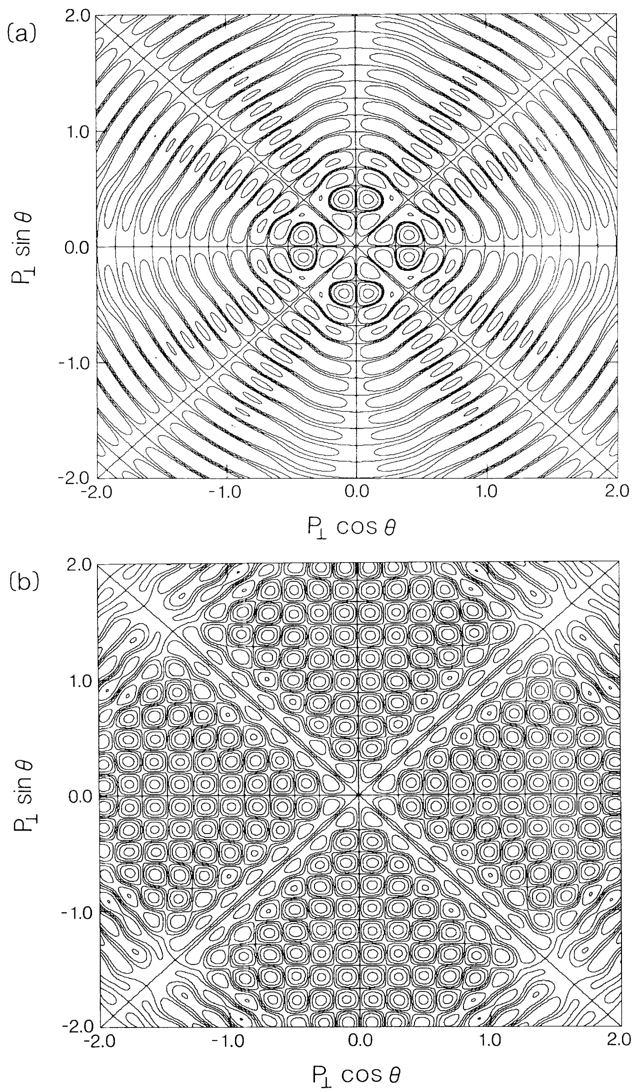


FIG. 2. Stochastic web. The wave packet comprises sine waves with $n_1 \cos\alpha_1 = 1$, $k_{\perp i} = k_{\perp j}$, $\Phi_{0i} = \Phi_{0j}$, $\Delta l = 4$ and $\omega_1 = 4\Omega$. (a) Two waves. (b) Ten waves. The structure for the case where the wave packet comprises cosine waves would be similar to what is shown here, but the elongated regions extending to large energies would cover a broader area.

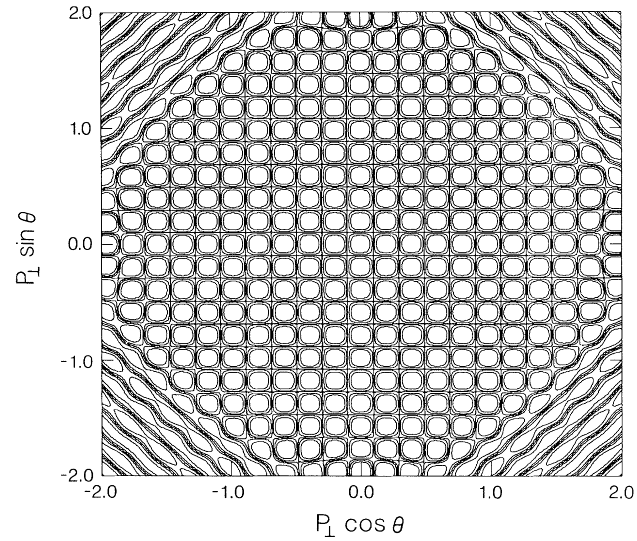


FIG. 3. Effect of $l=0$ wave. Parameters are the same as in Fig. 2(b) except that $\omega_1 = 0$ with $\Phi_{0i} = 2\Phi_{01}$ for $i \geq 1$. The squared region has the web structure studied by Zaslavskii *et al.* (Ref. 7). The region beyond the squares is due to the finite number of waves.

Noting that $\rho \propto \gamma_0$, it can be easily shown that for $\alpha_i \sim 90°$ the left-hand side of this inequality increases faster with energy than the right-hand side. Therefore, for a given amplitude, although the system is intrinsically degenerate at low energies, it quickly becomes accidentally degenerate at higher energies, which in turn results in the destruction of the web. If, on the other hand, $n_i \cos\alpha_i = 1$, the system would remain intrinsically degenerate at all energies.

In order to examine the phase-space structure, we again plot contours of constant-$\Delta\bar{H}$ surfaces. Such plots are more time efficient than particle runs and reproduce the phase-space structure very well as long as the wave amplitudes are well below the stochasticity threshold. They are preferable to the solution of the mapping equations where use is made of a $\delta$ function to simplify the equations of motion which in turn obscures the physical importance of the number of waves used in the wave packet. We emphasize that $\Delta\bar{H}$ in the case of an electrostatic wave packet has the same form as the nonrelativistic one describing the Zaslavskii *et al.* web. Thus, one would expect to get the same web structure in the relativistic case as in the nonrelativistic case.

There are, however, some differences arising from the fact that $l_i \geq 1$ in the relativistic system, whereas $l_i$ can range from $-\infty$ to $+\infty$ in the nonrelativistic case. Figure 2 illustrates the effect of the finite number of waves on the phase space. As $N$ is increased, the fixed points above a certain energy bifurcate and the large first-order islands of the one-wave case are broken up into smaller islands, with the bifurcation starting at small energies and spreading to larger energies as $N$ is increased. Once the bifurcations start at a given point in phase space, they quickly saturate as a function of $N$ and stochastic layers are formed in that region [Fig. 2(b)]. We distinguish three regions in Fig. 2(b): (1) A region covered by squarelike structures; these are similar to the nonrelativistic stochastic web. (2) The X-shaped large-scale islands which allow particle acceleration on a faster time scale than that through region (1). These structures are present as long as the $l_i = 0$ term in Eq. (7) is absent. This can happen if the electrostatic potential is composed of sine waves or if it is composed of cosine waves with the $l=0$ wave either absent or not satisfying the resonance condition. Figure 3 illustrates the change in the

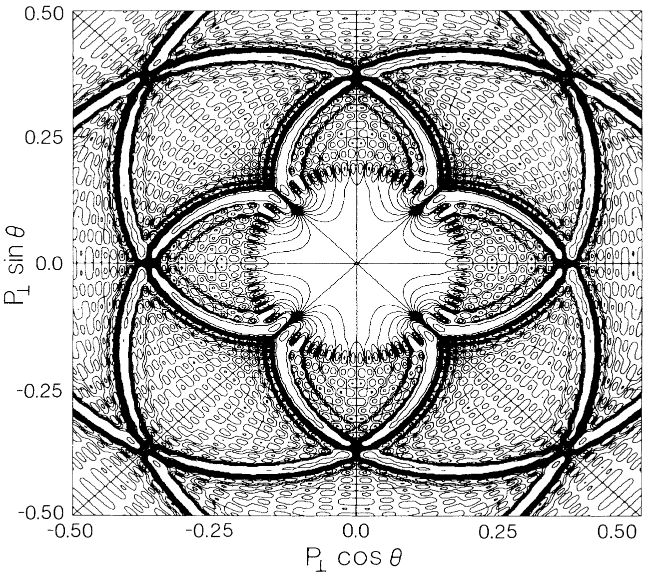


FIG. 4. The stochastic web showing the effect of unequal $k_{\perp i}$'s at a fixed angle ($\alpha_i = \alpha_j$).

web structure had the $l=0$ resonance been present. Note, however, that $l=0$ resonance cannot be satisfied in the relativistic intrinsic degenerate case. (3) A third region which is the part of phase space extending beyond regions (1) and (2) and looks much like the phase space of the one-wave case.

The symmetry of the web is determined by the frequency separation between consecutive harmonics in the wave packet, $\Delta\omega = (\Delta l)\Omega mc^2/H_0$, and the extent of the web in momentum space is determined by $N$. If $N \to \infty$, regions (1) and (2) extend to infinite energy, in contrast to the $\alpha = 90°$ case where the web structure is limited to a small energy even in the presence of an infinity of waves. We can observe immediately one of the key features of the relativistic treatment: In contrast to the nonrelativistic case one does not need many waves to reach large energies through the stochastic web. This is due to the fact that the relativistic trapping widths are several orders of magnitude larger than their nonrelativistic counterparts. If the waves have the same electric field rather than electrostatic potential amplitudes, then the effect of addition of higher frequencies is reduced and the phase space resembles that of a smaller-$N$ case.

If one relaxes the condition $k_{\perp i} = \text{const}$, the phase space acquires a very different structure (Fig. 4). One can still observe a small-scale structure, reminiscent of the Zaslavskii *et al.*[7] web. However, this web is disected by a set of elongated islands that inhibit considerably the penetration of particles to large energies. This large-scale structure is four symmetric due to the fact that the spacing between $l_i$'s is a multiple of 4. It is a remnant of the large first-order islands of the one-wave case that did not break into smaller ones after the addition of more waves but were deformed into almost impenetrable separations between parts of the phase space. For $k_{\perp i} \neq k_{\perp j}$ and a random distribution of $\alpha_i$'s, the phase space has a noiselike structure and no web is present.

In summary, the necessary condition for the formation of the stochastic web in the problem of wave-particle interaction in a strong magnetic field is for the system to be intrinsically degenerate [Eq. (7)]. For $\alpha = 90°$, the zero-order resonance condition reads $\omega_i = l_i\Omega/\gamma_0$ and it is clear that the relativistic mass increase destroys the resonance to zero order, rendering the system accidentally degenerate. However, when $n_i \cos\alpha_i \to 1$ and $\omega_i \to l_i \Omega mc^2/H_0$, the resonance surface and the $H_0$ surface become coincidental and the change with action in $\gamma_0$ is balanced with the change in $P_z$. This allows the formation of a web in phase space to arbitrary energies. We also showed that the relativistic stochastic web is obtained under more realistic conditions than the nonrelativistic case. Detailed application of these ideas will be presented elsewhere.

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