Comment on "Dynamical Symmetries of the Perturbed Hydrogen Atom: The van der Waals Interaction"

In a recent Letter[1] Alhassid, Hinds, and Meschede reported the dynamical symmetries and the analytic spectrum of the hydrogen atom in a generalized van der Waals potential $V=\gamma(x^2+y^2+\beta^2 z^2)$, where γ and β are constants. Here $\beta=\sqrt{2}$ corresponds to the instantaneous van der Waals potential and $\beta=0$ corresponds to the diamagnetic potential. They also found a universal adiabatic invariant and in particular dynamical symmetries for the three special choices $\beta=2$, 1, and $\frac{1}{2}$.

We wish to point out here some more interesting properties of this potential. If we limit our discussion for convenience to $L_z=0$ states and introduce semiparabolic coordinates[2] so as to avoid the Coulomb singularity problem, $\mu=(r+z)^{1/2}$, $\nu=(r-z)^{1/2}$, then the classical (dimensionless) Hamiltonian

$$H=p^2/2-1/r+\gamma(x^2+y^2+\beta^2 z^2)=E$$

of the problem becomes that of a set of two coupled sextic anharmonic oscillators,

$$H=\tfrac{1}{2}(P_u^2+P_v^2)+\tfrac{1}{2}(u^2+v^2)+A(u^6+v^6)+B(u^4v^2+u^2v^4)=2\epsilon\,, \quad (1)$$

where $u=(-2E)^{1/4}\mu$, $v=(-2E)^{1/4}\nu$, $A=\gamma\beta^2/4$, $B=\gamma\times(1-\beta^2/4)$, and $\epsilon=1/(-2E)^{1/2}$.

By applying Painleve singularity analysis to the equations of motion of the above coupled-oscillator system[3] we locate at least three integrable cases: (i) $B=0$, (ii) $B=3A$, and (iii) $B=15A$, besides the trivial case $A=B=0$. In fact, for the $B=0$ case the system decouples into two independent sextic oscillators and in the original variables this case corresponds to $\beta=2$, γ arbitrary. For $B=3A$ in (1) we have the second integral of motion $I_2=(uP_v-vP_u)^2$ and this case corresponds to $\beta=1$, γ arbitrary, which is the spherically symmetric case in the original variables. For the $B=15A$ case we have

$$I_2=[P_uP_v+uv+6A(u^4+v^4)uv+20Au^3v^3]$$

as the second integral of motion which corresponds to $\beta=\frac{1}{2}$, γ arbitrary. Thus the three integrable cases $B=0$, $3A$, and $15A$ ($\beta=2,1,\frac{1}{2}$) of our analysis correspond exactly to the special cases where dynamical symmetries exist as pointed out by Alhassid, Hinds, and Meschede,[1] thereby bringing out the reason behind the existence of these symmetries.

The system of classical coupled oscillators (1) is simpler to analyze numerically. Integrating it using the Runge-Kutta-Gill fourth-order method, we plotted the Poincaré surface of section ($v=0, P_v>0$) of a single trajectory in the u-P_u plane and found that the system shows chaos-order-chaos transitional behavior when the value of B is varied. We fixed the value of A at $\frac{1}{6}$. For all negative values of B the system becomes fully chaotic. For $B=0$, we have regular behavior. For values of B in between 0 and 0.5 the invariant tori break into small islands, implying small-scale manifestation of chaos. For $B=0.5$ the system shows regular behavior again. For values of B in between 0.5 and 2.5 we again notice small-scale manifestation of chaos in the form of islands. For $B=2.5$ we again get regular behavior. For values of B in between 2.5 and 3.5 the tori again break into islands, while for all values of B greater than 3.5 the system shows fully chaotic behavior. Thus this problem may turn out to be an interesting physical system with which to study chaos-order-chaos behavior experimentally.

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