Ion Density Cavities Can Cause Nonlinear Plasma Oscillations to Peak

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In this Letter an exact solution for nonlinear cold-electron-plasma oscillations against a periodic ion background is found. The main difference with respect to the known solution corresponding to a uniform ion background is that, after a while, electron density peaks will appear. This phenomenon, which was foreseen in general terms by Dawson, has possible practical implications which are indicated briefly. A thermal limitation on the peaks is given.

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Periodic ion distributions are assumed in several plasma contexts. For example, anomalous resistivity in turbulent plasmas is not well understood.[1,2] When the weak-turbulence collision frequency is less than the electron cyclotron frequency, the electron motion is essentially one dimensional. For this case it proves difficult to obtain appreciable anomalous resistivity from the weak-turbulence theory.[3] To resolve this difficulty, a strong-turbulence theory was presented in Ref. 4. An array of ion density cavities was assumed.

In the auroral region, electrostatic ion cyclotron waves may also give rise to density cavities. Computer simulations have been looked at in this context.[5]

In computer simulations for plasmas with periodic ion density cavities it is usually assumed that the ion distribution remains constant in time. Simulations indicate the formation of strong electric field gradients. These have also recently been observed in the aurora.[6]

In this Letter we look at a simple exact solution for cold-electron-plasma oscillations against a periodic ion background, assumed constant on the time scale of the electron motion. We find the salient features of the above-mentioned observations and simulations,[5,6] most importantly a steepening of the electric field gradient caused by the presence of the ion cavities.

The problem of nonlinear cold-electron-plasma oscillations against a uniform ion background was solved many years ago by several authors simultaneously.[7-9] They found oscillations with the electron plasma frequency $\omega_p = (4\pi n_0 e^2/m_e)^{1/2}$. Thus the frequency was amplitude independent. In general (unless the initial conditions were especially tailored such that at the onset the amplitude of the oscillations was at least $\frac{1}{2}$ of the background density) the electron density was not found to exhibit explosive behavior. A good reference for this solution is Chapter 3 of Davidson's book.[9] However, Ref. 7 gives a heuristic argument for wave breaking (infinite densities) when the ion background is not uniform.

Plasma oscillations under the influence of an applied sinusoidal field have been considered by several authors.[10] The forcing term is usually assumed small. Enhanced electron density was obtained. The situation beyond breaking was considered by mathematical methods that are not universally accepted (such as altering Poisson's equation, changing the sign of the density when it becomes negative, etc.). Nevertheless, a theory of pump energy conversion to the plasma is obtained in the first of these references.

In a recent paper,[11] Infeld and Rowlands generalized the exact cold-plasma solution to a relativistic plasma. For small ω_{pe}/ck, where $2\pi/k$ is the wavelength, the nonlinear oscillations described above were of course recovered for a while. However, after a number of oscillations proportional to $(ck/\omega_{pe})^2$, the density was found to peak. Thus general initial conditions were found to lead, after a while, to infinite electron density at a point (in practice a sequence of electron density bursts was to be expected). Large electron density was of course accompanied by large electric field gradients. Whether the time scale required for these "relativistic bursts" was practical or not depended on the parameter $(ck/\omega_{pe})^2$. The only exception to the above behavior results from the very special initial conditions that lead to a Bernstein-Greene-Kruskal wave.

In the present paper we solve the problem of nonrelativistic, cold-electron-plasma oscillations against a fixed periodic ion density background. In view of the above considerations, it is physically interesting that electron density bursts are found here too, although not surprising in view of Ref. 7. Thus an exact nonlinear solution, of interest in its own right, seems to be useful for comparison with computer simulations and, perhaps, with auroral observations.

In the calculation we assume the ion density to be time independent and space periodic. We take

$$n_i(x,t) = n_0[1+\alpha\cos(kx)], \quad \alpha < 1\,, \tag{1}$$

and a uniform initial electron density

$$n_e(x,0) = n_0\,. \tag{2}$$

This will give an initial electric field. To find it we use Poisson's equation

$$\partial E/\partial x = 4\pi e(n_i - n_e), \quad (3)$$

yielding

$$E(x,0) = 4\pi e n_0 \alpha k^{-1}\sin(kx). \quad (4)$$

Here a possible uniform component of E is taken to be zero, as is the initial electron velocity $v_e(x,0)$. The field (4) created by local charge inbalance will drive the system in a nonlinear mode.

The equations for the electron density n_e and velocity v_e are, in the fluid model of a cold plasma,

$$\partial n_e/\partial t + \partial(n_e v_e)/\partial x = 0, \quad (5)$$

$$\partial v_e/\partial t + v_e\,\partial v_e/\partial x = -eE/m_e. \quad (6)$$

Equations (3), (5), and (6) give a complete description of the plasma.

We now introduce Lagrangian coordinates x_0, τ,[10-12] and an auxiliary function ψ:

$$\psi = \int_0^\tau v_e\,d\tau, \quad x_0 = x - \psi, \quad \tau = t. \quad (7)$$

The coordinate x_0 follows a fluid element in its motion. Our equations simplify in the new coordinates to

$$(\partial/\partial\tau)[n_e(1+\partial\psi/\partial x_0)] = 0, \quad \partial\psi/\partial\tau = v_e, \quad (8)$$

$$\partial v_e/\partial\tau = -eE/m_e, \quad (9)$$

$$\partial E/\partial\tau = +4\pi e n_i v_e. \quad (10)$$

The first equation can be integrated to yield

$$n_e = n_e(x_0,0)/[1+\partial\psi/\partial x_0]. \quad (11)$$

The second and third equations, (9) and (10), can be combined to yield

$$\partial^3\psi/\partial\tau^3 + \omega_{pe}^2[1+\alpha\cos(k\{x_0+\psi\})]\partial\psi/\partial\tau = 0. \quad (12)$$

Upon integrating twice and using the initial conditions, we obtain

$$(\partial\psi/\partial\tau)^2 = -\omega_{pe}^2\psi^2 + 2\alpha\omega_{pe}^2 k^{-2}[\cos(k\{x_0+\psi\}) - \cos(kx_0)]. \quad (13)$$

We can rescale the variables, introducing $\phi = k\psi$, $\bar\tau = \omega_{pe}\tau$, $\bar x = kx_0$, to obtain an exact solution in simple parametric form. It is, in terms of n_e and Eulerian variables x,t, interpreting the integral in (15) such that t increases monotonically (see also Fig. 1),

$$kx = \bar x - \phi, \quad (14)$$

$$\omega_{pe}t = \int_0^\phi d\phi'/[2\alpha\{\cos(\bar x+\phi') - \cos\bar x\} - \phi'^2], \quad (15)$$

$$n = n_0/[1+\partial\phi/\partial\bar x], \quad (16)$$

$$\partial\phi/\partial\bar x = \alpha[2\alpha\{\cos(\bar x+\phi) - \cos\bar x\} - \phi^2]^{1/2}\int_0^\phi \frac{\sin(\bar x) - \sin(\bar x - \phi')}{[2\alpha\{\cos(\bar x+\phi') - \cos\bar x\} - \phi']^{3/2}}\,d\phi'. \quad (17)$$

This is indeed an exact solution in parametric form $x = x(\bar x,\phi)$, $t = t(\bar x,\phi)$, $n = n(\bar x,\phi)$. Note how very much simpler it was to obtain than those found by the ubiquitous inverse scattering method. However, as is often the case, some features of the solution are more simply seen by methods other than plotting the exact solution (14)-(17).

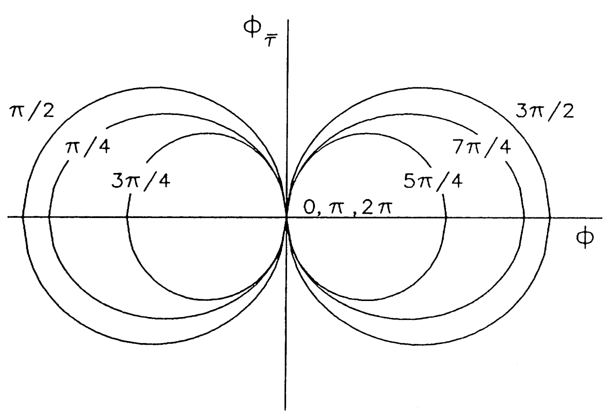

FIG. 1. Phase-plane solution curves $(\phi_{\bar\tau},\phi)$ for chosen values of kx_0 indicated in the figure. Here $\alpha = 0.3$.

Two features of this exact solution will now be needed: (1) The motion of each individual fluid element, labeled by one x_0, is periodic. (2) The period of the motion T is a function of x_0 (this is in contradistinction to the cold-plasma, uniform-ion-background case, for which $T = 2\pi/\omega_{pe}$).

Both features are seen from the figures. Figure 1 shows curves in phase space corresponding to the solution and given by (13) for chosen x_0. If all curves for kx_0 between 0 and 2π were drawn, they would fill in the two regions. (These curves are not quite elliptical.). To follow the motion of one fluid element, labeled by given x_0, through one period, we go around the corresponding phase curve once. Both $\phi_{\bar\tau}$ and ϕ are thus periodic functions of time provided T is finite. We now simply calculate $T(x_0)$ numerically by taking (15) up to the zero of the denominator. The result is shown in Fig. 2. Indeed T is seen to be a (nonconstant) finite function of x_0. As stated above, ϕ, and hence ψ, is a periodic function of τ,

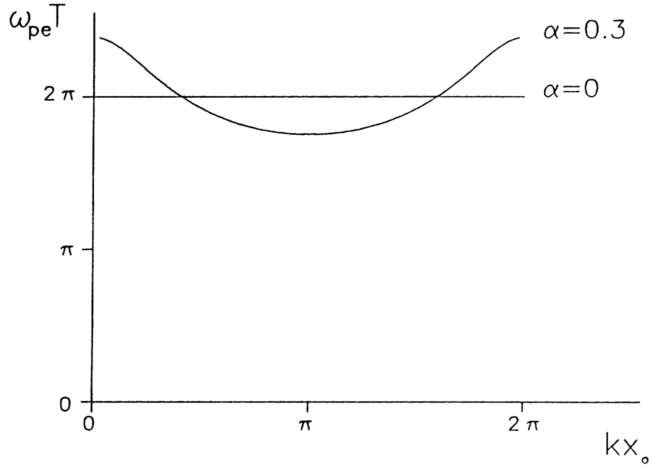

FIG. 2. Each fluid element, labeled by x_0, oscillates with its own period T. Dependence of T on x_0 for $\alpha=0.3$.

but the period depends on x_0. We can write this as

$$\psi=\psi(x_0,\tau/T)\,, \tag{18}$$

where the period in the second variable is now one. Thus

$$\partial\psi/\partial x_0=\partial\psi/\partial x_0-(\tau/T^2)(\partial T/\partial x_0)\psi'\,, \tag{19}$$

where the prime denotes differentiation with respect to the second argument. As ψ' goes through both signs, the secular component will sooner or later cause the denominator in (16) to vanish. Thus, for some finite time τ, n_e becomes infinite. We see from (3) that this will also cause $\partial E/\partial x$ to become infinite. Without the periodic ion background, $\partial T/\partial x_0$ is zero and this explosive behavior is not seen. [The secular component can also be seen in (17).] This is the effect mentioned in Ref. 7, and our general Eq. (17) gives Eq. (22) of that reference.

Figure 3 shows lines of constant x_0 in (x,t) space. Bursts appear whenever these lines coalesce.

Whenever the model predicts infinite n_e it must give way to a more complete physical description. Presumably this would lead to large but finite n_e and $\partial E/\partial x$, which would then subsequently relax. After a while, our model could once again be used with different initial conditions and would give a second burst. Thus we predict that a periodic ion background will cause a sequence of bursts in n_e and $\partial E/\partial x$ to appear in the plasma. This could be an example of intermittency in plasma physics.

A small-amplitude calculation, including a small electron thermal effect in (6), but no longer exact as the above considerations were, gives a *finite-amplitude* burst in n_e after a time T_b:

$$T_b=2\pi K(1/\sqrt{2})/k\lambda_D(\gamma\alpha)^{1/2}\,.$$

Here K is the complete elliptic integral, λ_D is the Debye length, $\gamma=c_p/c_v$, and α is now assumed small. This formula was derived by expanding in both the electron temperature (in $k\lambda_D$) and in α; see the penultimate reference in Ref. 12 for a somewhat similar expansion.

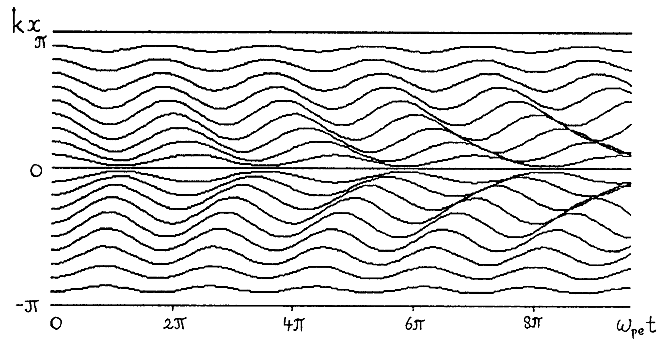

FIG. 3. Lines of constant x_0 for $\alpha=0.3$. These lines are seen to coalesce at (roughly) $\omega_{pe}t=7\pi$, leading to bursts.

As mentioned above, the electron density no longer has to become infinite and its maximum, attained at $t=T_b$, is

$$n_{e\,\max}=n_0/(1-\alpha^{3/2}k^{-1}\gamma^{-1/2}\lambda_D^{-1})\,.$$

This is finite and positive when $\lambda_D>\alpha^{3/2}/k\gamma^{1/2}$. Thus to limit the bursts the electron temperature must exceed a critical value. (On the other hand, if it is too large the bursts will not appear at all.)

We suggest that further observations and simulations should be aimed at finding a definite and widespread correlation between the occurrence of ion cavities and electric field steepening as predicted qualitatively by Dawson and quantitatively by our exact solution. In the small-α limit our formula for T_b could also be checked.

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