Information Theoretical Characterization of Turbulence

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A method of describing turbulence in terms of dynamical connectivity in the wave-number space is proposed. The connectivity is quantified by the information theoretical quantities, i.e., mutual information and cross information flow rate. The method is applied to the analysis of two simple examples of turbulence in one spatial dimension. Although the examples have quite different physical origins, the information structures of the wave-number space turn out to be quite similar: The wave-number space consists of several regions generating information in different ways, and how information flows between these regions reflects the dynamical structure of turbulence.

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Much effort has been made to describe the complex dynamical behavior we call chaos. Several methods have been proposed and applied to characterizing chaotic behavior quantitatively.[1] However, all methods proposed so far are only applicable when the dimension of the chaotic attractor is not very large. In order to reduce the dimension of chaos, experiments are usually made in carefully controlled *artifical* environments. However, most chaotic behavior observed in *natural* environments will be more or less high dimensional as typically exemplified by fluid turbulence. Unfortunately, we do not have an efficient method to describe high-dimensional chaos which seems to have a more natural occurrence. The aim of the present paper is to show an attempt to characterize very-high-dimensional chaos accompanied by homogeneous spatiotemporal variation.

For spatially homogeneous turbulence the wave-number space (Fourier space) is of fundamental importance: Mathematically, the Fourier basis can confine the chaotic attractor most efficiently in the following sense. Consider an orthogonal basis $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n,\ldots\}$. Let S_n be the subspace spanned by $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$. If the motion of the system represented by the state vector $\mathbf{R}$ is homogeneous in the real space, then the deviation of $\mathbf{R}$ from its projection onto S_n can be made minimum on average by choosing $\{\mathbf{e}_1,\mathbf{e}_2,\ldots\}$ to be the Fourier basis. The physical reason why we regard the wave-number space as essential is as follows. A homogeneous turbulent motion is first induced by an instability of Fourier modes in a restricted region in the wave-number space and it is then sustained by the nonlinear mode-mode coupling which suppresses the instability by forcing the energy to flow toward the higher-wave-number region. Indeed, recent study of very-high-dimensional chaotic motion accompanied by strong spatiotemporal mixing reveals that the scheme of motion in the wave-number space depends on whether Fourier modes contribute actively to chaotic motion or not.[2] The above considerations say that the connectivity of dynamics in the wave-number space, which directly reflects how chaotic information is generated and transmitted, is of fundamental importance to the description of turbulent motion. Our idea is to describe turbulent motion in terms of the dynamical connectivity in the wave-number space, quantified by information theoretical quantities.

How can we characterize the connectivity in the wave-number space in terms of information theoretical quantities? For the sake of simplicity we consider spatially one-dimensional stationary turbulence which is homogeneous in real space. Suppose $\epsilon(k,t)$ (t, time) to be the time sequence of a physical quantity defined at the position k in the wave-number space. To introduce information theoretical quantities we first quantize the value taken by $\epsilon(k,t)$ into N levels. Simultaneously, we regard time as discrete steps with the interval Δ. Then we can represent the time sequence $\epsilon(k,t)$ observed during a finite period $T=n\Delta$ by a discretized sequence $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)$, where α_i is one of the N quantized levels. Let $P(\boldsymbol{\alpha})$ be the probability that one of the sequences $\boldsymbol{\alpha}$ in the N^n possible states is realized. Then $H^{(n)}(k)=-\sum_{\boldsymbol{\alpha}}P(\boldsymbol{\alpha})\ln P(\boldsymbol{\alpha})$ is the n-step Shannon entropy for the sequence $\epsilon(k,t)$. Next, consider a combined set of two time sequences $\epsilon(k_s,t)$ and $\epsilon(k_r,t)$ at two different positions k_s and k_r in the wave-number space. Suppose $P(\boldsymbol{\alpha}_s,\boldsymbol{\alpha}_r)$ to be the joint probability that $\epsilon(k_s,t_1)$ and $\epsilon(k_r,t_2)$ take the sequences $\boldsymbol{\alpha}_s$ and $\boldsymbol{\alpha}_r$ for $0<t_1\le n\Delta$ and $\tau<t_2\le\tau+n\Delta$, respectively. Then the n-step Shannon entropy for the joint process is defined by

$$H^{(n)}(k_s,k_r\,|\,\tau)=-\sum_{\boldsymbol{\alpha}_r,\boldsymbol{\alpha}_s}P(\boldsymbol{\alpha}_r,\boldsymbol{\alpha}_s)\ln P(\boldsymbol{\alpha}_r,\boldsymbol{\alpha}_s)\,.$$

The mutual information (MI) which represents the information carried in common by the two sequences

$\epsilon(k_s,t_1)$ and $\epsilon(k_r,t_2)$ is defined in terms of the two Shannon entropies,[3,4]

$$I^{(n)}(k_s,k_r\,|\,\tau)=H^{(n)}(k_r)+H^{(n)}(k_s)-H^{(n)}(k_s,k_r\,|\,\tau)\quad(\geq 0)\,. \tag{1}$$

The MI takes the minimum value zero if and only if the two sequences $\epsilon(k_s,t_1)$ and $\epsilon(k_r,t_2)$ are statistically independent. It measures the amount of connectivity between two sequences. On the other hand, the rate of increase of the MI

$$K(k_s{:}k_r)=\lim_{\Delta\to 0}\{\lim_{n\to\infty}[I^{(n)}(k_s,k_r\,|\,\tau)-I^{(n-1)}(k_s,k_r\,|\,\tau)]/\Delta\}\quad(\geq 0)\,, \tag{2}$$

which no longer depends on the time lag τ, is called the cross information flow rate and is the rate of common information generated in the two times sequences per unit time.[5]

Using these information theoretical quantities we demonstrate how the connectivity in the wave-number space is characterized for simple examples of turbulence. The model systems we examine exhibit spatially one-dimensional turbulence with a very-high-dimensional chaotic attractor. The first example is the delay-differential (DD) model of optical turbulence,[2]

$$d\psi(t)/dt=-\psi(t)+\mu f(\psi(t-l))\,, \tag{3}$$

with $f(\psi)=\cos(\psi-\phi_0)$ (ϕ_0, constant) and l standing for the system size. The DD equation can be regarded as a discretized mapping rule from a "spatial" pattern at time n (n, integer), i.e., $\psi(x,n)\equiv\psi(nl+x)$ $(0\leq x<l)$ to $\psi(x,n+1)$. Hereafter we follow this description. The second example is the Kuramoto-Sivashinsky (KS) model of chemical turbulence,[6]

$$\psi_t(x,t)=-\psi_{xx}(x,t)-\psi_{xxxx}(x,t)-\psi_x(x,t)\psi(x,t)\,, \tag{4}$$

with the periodic boundary condition $\psi(x+l,t)=\psi(x,t)$. Let $\psi(k,t)$ be the Fourier transform of $\psi(x,t)$. Then the Fourier spectrum, i.e., the long-time average of $|\psi(k,t)|^2$, observed for the turbulent states of our system has the following features: It has a noticeable peak at $k=k_0$ ($k_0=0$ for DD and $k_0\approx 1/\sqrt{2}$ for KS) and decays exponentially beyond a certain wave number[2,6] k_d ($>k_0$).

In applying our information theoretical method outlined above it would to be quite natural to choose $|\psi(k,t)|^2$ as $\epsilon(k,t)$. However, instead of $|\psi(k,t)|^2$ itself, we choose an average of $|\psi(k,t)|^2$ over a finite band $k\leq k'\leq k+\Delta k$, i.e.,

$$\epsilon(k,t)=\sum_{k'=k}^{k+\Delta k}|\psi(k',t)|^2\,,$$

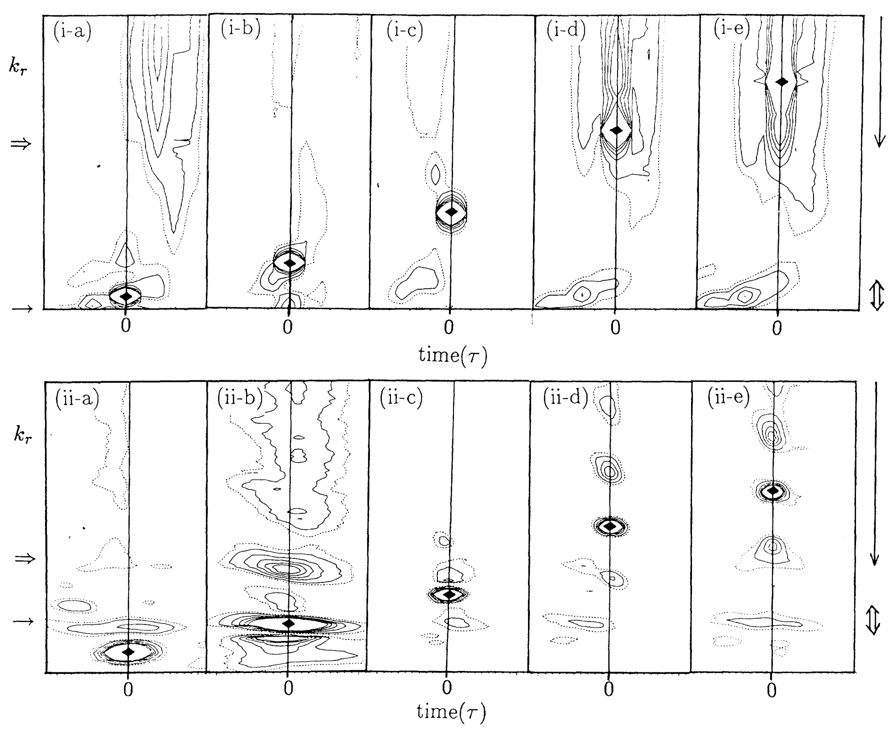

FIG. 1. Information maps (i) DD model ($\mu=1.3$, $l=40$) and (ii) KS model ($l=128$), where ⟹ and → indicate k_d and k_0, respectively, and ◆ corresponds to the position of the source k_s at $\tau=0$. The C and D regions are indicated by ⟺ and ↓, respectively.

because we are concerned with the average behavior in the vicinity of a given k.

In Figs. 1(i) and 1(ii) we show the "information map," i.e., the contour plot of MI $I^{(1)}(k_s,k_r\,|\,\tau)$, obtained for the turbulent states of the DD and KS models, respectively. We show how the pattern of MI over the "receiver's" spacetime (k_r,τ) changes as the "source" k_s is moved toward the higher-wave-number side in the order (a), (b), (c),.... The positions of the two characteristic wave numbers k_0 and k_d are indicated by arrrows. Although the two models have quite different physical origins their information structures are quite similar: There are three regions of k_s in which the information map exhibits quantitatively different patterns. We call these three regions the core region, the intermediate region, and the dissipative region, respectively. The core (C) region forms a narrow band surrounding $k_s=k_0$, whereas the dissipative (D) region corresponds to the high-wave-number domain roughly $k_s>k_d$. Thus C and D regions are separated, and the remainder of the wave-number space is occupied by the intermediate (I) region.

Now let us read the information map. When k_s is in the C region the pattern of MI spreads most widely in the wave-number space [Figs. (i-a), (ii-b)]. In particular it spreads over the D region with large amplitude. Moreover, for all k_r in the D region the maximum of MI appears with the same delay after the MI maximum at the source k_s. This implies that there is a flow of information from the C region to the D region. However, there is a "gap" region to which information does not flow from the C region. This region corresponds to the I region.

The strong connection between the C and D regions is more evident when k_s moves into the D region. A most remarkable characteristic for $k_s\in$ (D region) is that the MI pattern is delocalized in the D region [panels (d),(e) in Figs. (i) and (ii)]. Therefore, the motions in the D region are strongly correlated. The MI has significant magnitude in the C region as well. The time when the MI has the maximum is the same for all k_r in the D region and it is delayed from the time when the MI takes the maximum at $k_r\in$ (C region). This means that the influence of the dynamical events occurring in the C region propagates and emerges in the D region after a delay time. In short, the origin of motion in the D region is in the C region. In the case of KS the MI is periodic with period k_0 for $k_r\in$ (D region), which indicates nonlinear coupling of k_s with the source k_0. Thus the D region does not generate information by itself and belongs to the "exterior" of the attractor.[2] The strongly correlated emission of intermittent bursts observed in the D region[2] is responsible for such a behavior.

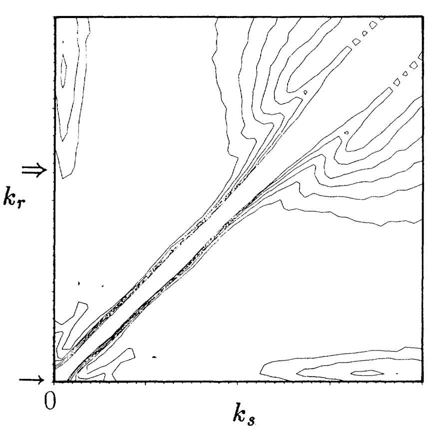

FIG. 2. Contour plot of the cross information flow rate. See the text.

In contrast to the D region the MI pattern is localized when k_s is in the I region [(b),(c) in Fig. (i) and (a),(c) in Fig. (ii)]. The I region is isolated from both the D and C regions. These features are very noticeable for the DD model, and a similar behavior is seen for the KS model as well. The dynamics of different modes in the I region clearly generate information independently. This fact implies that the C region corresponds to the subspace which is actively generating chaotic information. Indeed the Lyapunov spectrum analysis reveals that the "interior" of the attractor, i.e., the subspace spanned by the Lyapunov vectors whose numbers are less than the Lyapunov dimension, is contained in the subspace spanned by the modes in the C and I regions.[2] Thus the I and C regions roughly correspond to the interior of the attractor.

The characterization by the cross information flow rate (CIFR) introduced by Eq. (2) provides a more compact summary of the information structure of the wave-number space. We show in Fig. 2 a plot of contour lines of $K(k_s\!:\!k_r)$ $[=K(k_r\!:\!k_s)]$ obtained for the DD model. The relations between the three regions discussed above are quite obvious from this plot. A defect of this method is, however, that it cannot describe the direction of the information flow, which is easily judged from the MI map. Thus the characterizations by the MI and by the CIFR complement each other. These methods can be applied to analyze experiments in which Fourier analysis data may be obtained. Dye-laser optical turbulence is a possible candidate.[7] It is of interest how the cascade propagation of energy and/or vorticity in fluid turbulence is described by our method. The study along this line is being developed.

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[1]See, for example, articles in *Dimensions and Entropies in Chaotic Systems,* edited by G. Mayer-Kress (Springer-Verlag, Berlin, 1986).

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