Derivation of Anomalous Commutator and Jacobian from Very General Conditions

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(Received 28 October 1988)

Using only the equation of motion and the canonical commutator of anomalous gauge theories, we show clearly the relation between the Schwinger-Jackiw-Johnson (SJJ) term, anomalous Jacobian, and the current-divergence anomaly by obtaining both the SJJ term and the anomalous Jacobian from the current-divergence anomaly.

PACS numbers: 11.15.Ex, 11.30.Rd

Recently, anomalous gauge theories have attracted much interest.[1] It is conjectured cohomologically that in an anomalous gauge theory the Schwinger-Jackiw-Johnson (SJJ) term should appear in the algebra of the Gauss-law constraints.[2] This conjecture is confirmed,[3,4] by Jo and others, through the Bjorken-Johnson-Low (BJL) method. Their calculations involve manipulation such as perturbative theory with quantized gauge fields. Meanwhile, a number of authors[5] propose that it is plausible to introduce a Jacobi-identity violation as an obstacle to the two-cocycle condition of the SJJ term. It is well understood cohomologically that when nontrivial boundary conditions are assumed, a nonvanishing Jacobian arises.[6,7]

In the present Letter we give a very general argument[8] in the fixed-time Hamiltonian approach to derive both the SJJ term and the anomalous Jacobian. We assume only the canonical equation of motion and the canonical commutators and allow the (source) currents to have an anomaly. Without going into the detailed regularization and dynamics, we show the relation between the SJJ term, the anomalous Jacobian, and the current-divergence anomaly by obtaining both the SJJ term and the anomalous Jocobian from the current-divergence anomaly.

(1) Anomalous Ward identity: $\partial_0 G^a(x) = [D_\mu J^\mu(x)]^a$.—In Weyl gauge, the equation of motion of gauge fields interacting with sources is

$$\begin{aligned} \partial_0 E_i^a(x) &= J_i^a(x) + [D_j F^{ji}(x)]^a , \\ E_i^a(x) &= -\partial_0 A_i^a(x) . \end{aligned} \tag{1.1}$$

It is easily shown that the Gauss-law constraint, $G^a(x) = J_0^a(x) + [D_i E^i(x)]^a$, satisfies the anomalous Ward identity

$$\partial_0 G(u) = \mathcal{A}(E,A;u) , \tag{1.2}$$

where

$$\begin{aligned} G(u) &= \int_{\mathbb{R}^3} d^3x\, u^a(x) G^a(x) \quad (u = t^a u^a) , \\ \mathcal{A}(E,A;u) &= \int_{\mathbb{R}^3} d^3x\, u^a(x) \mathcal{A}^a(x) , \end{aligned} \tag{1.3}$$

and $\mathcal{A}^a(x) = [D_\mu J^\mu(x)]^a$ is the anomaly in Weyl gauge; E and A are matrix-valued one-forms. In general, $\mathcal{A}$ is linear in the variable u, and it is a polynomial function in E and A and their derivatives with smooth coefficients.

The occurrence of the anomaly in (1.1) means that the constraint $G(u)$ is not a conservation quantity. When we impose on the physical state the Gauss-law condition at an initial time, this condition is not satisfied at a later time. Therefore, we cannot consistently quantize the anomalous gauge theory in the conventional ways.[1,9]

(2) Evaluation of the SJJ term.—Suppose that

$$[G(u), G(v)] = G([u,v]) + W(A;u,v) , \tag{2.1}$$

where W is the SJJ term; W is antisymmetric and bilinear in the variables u and v, and it is a polynomial function of A. From (2.1) one can obtain the consistency condition involving the anomaly and SJJ term,

$$\partial_0 W(A;u,v) = [G(u), \mathcal{A}(E,A;v)] - [G(v), \mathcal{A}(E,A;u)] - \mathcal{A}(E,A;[u,v]) . \tag{2.2}$$

In the first two terms the constraint $G(u)$ operates on $\mathcal{A}(E,A;u)$ by an operation on the arguments E ($\equiv t^a E^a \equiv t^a E_\mu^a dx^\mu$) and A ($\equiv t^a A^a \equiv t^a A_\mu^a dx^\mu$),

$$\begin{aligned} [G(u), \mathcal{A}(E,A;v)] &\equiv [G(u),E]^a \frac{\partial}{\partial E^a} \mathcal{A}(E,A;v) + [G(u),A]^a \frac{\partial}{\partial A^a} \mathcal{A}(E,A{:}v) \\ &\equiv \frac{d}{dt} \mathcal{A}(E + t[G(u),E], A; v) \Big|_{t=0} + \frac{d}{dt} \mathcal{A}(E, A + t[G(u),A]; v) \Big|_{t=0} . \end{aligned} \tag{2.3a}$$

Using the following canonical commutators

$$[E_i^a(x),A_j^b(y)]=\delta_{ij}\delta^{ab}\delta^3(x-y),\quad [A_i^a(x),A_j^b(y)]=0\,,$$

we have

$$[G(u),A]=Du\quad (D=d+[A,\,])\,, \tag{2.3b}$$

$$\begin{aligned}[G(u),E]&=[G(u),-\partial_0 A]=-\partial_0[G(u),A]+[\partial_0 G(u),A]\\ &=-\partial_0(Du)+[\mathcal{A}(E,A;u),A]=[E,u]+[\mathcal{A}(E,A;u),A]\,.\end{aligned} \tag{2.3c}$$

By (2.3a)-(2.3c), we obtain

$$\begin{aligned}[G(u),\mathcal{A}(E,A;v)]=&\frac{d}{dt}\mathcal{A}(E+t[E,u],A;v)\Big|_{t=0}\\ &+\frac{d}{dt}\mathcal{A}(E+t[\mathcal{A}(E,A;u),A],A;v)\Big|_{t=0}+\frac{d}{dt}\mathcal{A}(E,A+t\,Du;v)\Big|_{t=0}.\end{aligned} \tag{2.4a}$$

In the same way we have

$$\begin{aligned}[G(v),\mathcal{A}(E,A;u)]=&\frac{d}{dt}\mathcal{A}(E+t[E,v],A;u)\Big|_{t=0}\\ &+\frac{d}{dt}\mathcal{A}(E+t[\mathcal{A}(E,A;v),A],A;u)\Big|_{t=0}+\frac{d}{dt}\mathcal{A}(E,A+t\,Dv;u)\Big|_{t=0}.\end{aligned} \tag{2.4b}$$

Taking $\mathcal{A}$ to be the consistent anomaly in Weyl gauge

$$\mathcal{A}(E,A;u)=\frac{-i}{24\pi^2}\int_{\mathbb{R}^3}\mathrm{tr}\{E(dA\,u+u\,dA+\tfrac{1}{2}A^2u+\tfrac{1}{2}AuA+\tfrac{1}{2}uA^2)\}\,, \tag{2.5}$$

one can show the following identity:

$$\frac{d}{dt}\mathcal{A}(E+t[\mathcal{A}(E,A;u),A],A;v)\Big|_{t=0}=\frac{d}{dt}\mathcal{A}(E+t[\mathcal{A}(E,A;v),A],A;u)\Big|_{t=0}.$$

As a result, the right-hand side of (2.2) looks like the Becchi-Rouet-Stora-Tyutin transform of the consistent anomaly; the noncanonical terms cancel each other and only the canonical terms survive. The above derivation is very general. We only use the equations of motion and the canonical commutators; we avoid using the noncanonical E-E commutator; we do not go into the manipulation such as perturbative theory with quantized gauge fields. After long but straightforward work, we obtain from (2.2),

$$W(A;u,v)=\frac{-i}{48\pi^2}\int_{\mathbb{R}^3}\mathrm{tr}\{[u,v](dA\,A+A\,dA+A^3)+uA\,dvA-vA\,duA\}\,, \tag{2.6}$$

which is just the expression found by Jo[3] through the BJL method and differs from the expression originally predicted[2] only by some trivial terms.

(3) Evaluation of the anomalous Jacobian.— The W in the above section obeys an ordinary two-cocycle condition provided one has trivial boundary conditions for the gauge fields at spatial infinity. However, if there are persistent boundary effects, the W's modify the Jacobi identity of the constraints. To clarify this, we suppose the following triple commutator:

$$[G(u),[G(v),G(w)]]+\text{perm}=Z(A;u,v,w)\,, \tag{3.1}$$

where

$$[G(u),[G(v),G(w)]]\equiv G(u)[G(v)G(w)]-G(u)[G(w)G(v)]-[G(v)G(w)]G(u)+[G(w)G(v)]G(u)\,,$$

and Z is the anomalous Jacobian. In the following, we give a formal and algebraic derivation of the anomalous Jacobian Z. Differentiating (3.1) with respect to time, and replacing each time derivative of G with (1.2) and the commutators of G with (2.1), one obtains a consistency condition involving the anomaly, the SJJ term, and the anomalous Jacobian,

$$\begin{aligned}\partial_0 Z(A;u,v,w) = {} & [\mathcal{A}(E,A;u),W(A;v,w)] - [G([v,w]),\mathcal{A}(E,A;u)] + [G(u),[G(v),\mathcal{A}(E,A;w)]] \\ & - [G(u),[G(w),\mathcal{A}(E,A;v)]] + [\mathcal{A}(E,A;v),W(A;w,u)] - [G([w,u]),\mathcal{A}(E,A;v)] \\ & + [G(v),[G(w),\mathcal{A}(E,A;u)]] - [G(v),[G(u),\mathcal{A}(E,A;w)]] + [\mathcal{A}(E,A;w),W(A;u,v)] \\ & - [G([u,v]),\mathcal{A}(E,A;w)] + [G(w),[G(u),\mathcal{A}(E,A;v)]] - [G(w),[G(v),\mathcal{A}(E,A;u)]] \,. \end{aligned} \tag{3.2}$$

The above equation can be simplified by using (2.2); the final result reads

$$\begin{aligned} Z(A;u,v,w) = {} & [G(u),W(A;v,w)] + [G(v),W(A;w,u)] + [G(w),W(A;u,v)] \\ & + W(A;u[v,w]) + W(A;v,[w,u]) + W(A;w,[u,v]) \,, \end{aligned} \tag{3.3}$$

which becomes a consistency condition only involving the anomalous Jacobian and the SJJ term. After some computations, we get

$$Z(A;u,v,w) = \frac{-i}{48\pi^2}\int_{\partial\mathbb{R}^3} \mathrm{tr}\{A(du[v,w]+[v,w]du+dv[w,u]+[w,u]dv+dw[u,v]+[u,v]dw)\} \,. \tag{3.4}$$

If we assume the nontrivial boundary conditions on the vector potential at the spatial infinity, we can think of the space $\mathbb{R}^3$ as a unit disk with a smooth boundary $\partial\mathbb{R}^3$. Physically $\partial\mathbb{R}^3$ is the spatial infinity $\|\mathbf{x}\| \to \infty$. Then (3.4) is nonzero and a Jacobi-identity violation is recognized on the boundary $\partial\mathbb{R}^3$. Equation (3.4) coincides, apart from some trivial terms, with the appropriate anomalous Jacobian obtained by the present author in Ref. 6. However, in the present derivation the relation between the anomalous Jacobian and the current-divergence anomaly becomes more transparent.

The nonzero Jacobian Z defined solely on the boundary—a space of one dimension lower—obeys a fourfold constraint identity. The identity reads

$$\begin{aligned} & [G(u),[G(v),[G(w),G(x)]] + \mathrm{perm}] + \mathrm{perm} \\ & \quad = [G([u,v]),[G(w),G(x)]] + \mathrm{perm} - [G([v,w]),[G(x),G(u)]] + \mathrm{perm} + [G([w,x]),[G(u),G(v)]] \\ & \qquad + \mathrm{perm} - [G([x,u]),[G(v),G(w)]] + \mathrm{perm} + [G([u,w]),[G(x),G(v)]] \\ & \qquad + \mathrm{perm} - [G([v,x]),[G(u),G(w)]] + \mathrm{perm} \,, \end{aligned} \tag{3.5}$$

which is obtained by Hou and the present author[10] and Jo,[7] independently.

Discussion.—We have shown the relation between the SJJ term, the anomalous Jacobian, and the anomaly by obtaining both the SJJ term and the anomalous Jacobian from the anomaly. The nonvanishing Jacobian exists only on the boundary; if we require zero boundary conditions, then the Jacobian is zero. The SJJ term and the anomalous Jacobian are the other faces of the anomaly; that is, they are the realization of the same thing in different dimensions (i.e., in different formalism). This may be of some significance to the Faddeev and Shatashivili proposal[11] for consistently quantizing anomalous gauge theories. They suggest adding Wess-Zumino-Witten terms to the chiral fermionic action to cancel anomalies. It seems from the above discussions that when the current-divergence anomaly can be so canceled, then the Gauss-law commutator anomaly and the Jacobi-identity anomaly will also get canceled. The present derivation is certainly convincing because we use only the equations of motion and the canonical commutators of anomalous gauge theories.

I would like to thank Professor H. Y. Guo and Professor P. Wang for valuable discussions. Also I have benefited from the comments and correspondence of Professor J. Stasheff and Dr. S. G. Jo. Partial financial support was provided by the Chinese National Post-doctoral Foundation.

[1]For earlier reviews and references to the original literature see R. Jackiw, Center for Theoretical Physics Report No. CTP 1436, 1986 (to be published); S. Treiman, R. Jackiw, B. Zumino, and E. Witten, *Current Algebra and Anomalies* (Princeton Univ. Press, Princeton, NJ, 1985).

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