# Diffusion on Two Space and Time Scales

A. J. Lichtenberg and B. P. Wood
*Department of Electrical Engineering and Computer Sciences, and the Electronics Research Laboratory, University of California, Berkeley, California 94720*
(Received 27 February 1989)

Diffusion is explored in a two-dimensional phase space in which a connected separatrix layer (web) of intrinsic stochasticity bounds regions of regular motion (tiles). In the presence of weak extrinsic noise, if the web diffusion dominates, the noise slows the web diffusion rate; if the extrinsic diffusion dominates, the diffusion is enhanced. The diffusion is characterized by two space and time scales. For a local-equilibrium model analytic calculations agree well with numerical results.

PACS numbers: 05.45.+b, 05.40.+j, 05.60.+w

An important phenomenon in nonlinear dynamics is the diffusion through a divided phase space in the presence of extrinsic stochasticity. In two degrees of freedom the divided phase space generically separates into regions of connected intrinsic stochasticity and regions dominated by Kol'mogorov-Arnol'd-Moser (KAM) curves on which the intrinsic motion is regular.[1,2] For systems periodic in a phase variable, if phase-spanning KAM curves exist, the diffusion in the presence of extrinsic stochasticity can be characterized by regions of the action for which slow extrinsic stochasticity diffuses phase points across KAM curves in series with regions of the phase space in which the more rapid intrinsic diffusion prevails.[2,3] The global diffusion rate is the slow extrinsic rate but over a reduced phase space. In contrast, over a primarily connected intrinsically stochastic region with imbedded KAM surfaces (islands) the intrinsic rate is slowed by the extrinsic diffusion in and out of the island regions.[2,4] Usually this latter effect is of minor importance, but, as we shall see below, it is closely related to the phenomenon to be examined.

If a Hamiltonian system is constructed by resonantly perturbing a linear oscillator a new phenomenon appears in two degrees of freedom, that of a connected stochastic web.[5,6] Furthermore, if the perturbation is a periodic $\delta$ function with a low-order resonance, the resulting stochastic web is globally uniform over the phase space.[6] The tiling of the phase space into a connected stochastic web surrounding tile regions by KAM curves creates a simple topology for studying combined intrinsic and extrinsic stochasticity. Of particular interest is the interaction of the large space scale at which phase points cross separtrices between tiles and the small-space-scale diffusion within each web or tile.

An estimation of this two-space-scale diffusion was made in Ref. 6, where it was recognized that the diffusion rate should be proportional (in some limit) to the ratio of phase space of the web to the total phase space. However, an explicit calculation of the intrinsic web diffusion was not made, so that a complete treatment of the two-space-scale diffusion could not be made. In Ref. 7 we have explicitly calculated the global rate of web diffusion $D_{\rm web}$. We then reasoned qualitatively that the inclusion of extrinsic noise would lead to an asymptotic (in time) diffusion in which "the overall diffusion rate is a product of the global separatrix rate ($D_{\rm web}$) and the ratio of phase-space areas of intrinsic to extrinsic stochasticity." It is the purpose of this Letter to numerically confirm this assertion, to determine what is meant by asymptotic in terms of the two-space scales and two time scales involved, and to examine the diffusion on the nonasymptotic transition time scales.

We use the mapping representation for the kicked oscillator[6,7]

$$u_{n+1}=(u_n+K_a\sin v_n)\cos\alpha+v_n\sin\alpha\,,$$

$$v_{n+1}=-(u_n+K_a\sin v_n)\sin\alpha+v_n\cos\alpha\,,$$

where $\alpha=\omega T$ is the rotation angle of the oscillator between kicks, $K_a$ is the maximum kick amplitude, and $u$ and $v$ are the normalized velocity components. The mapping is composed of a product of two involutions, a step change in $u$ followed by a rotation, and is therefore measure preserving. At a resonance we have $\alpha=2\pi p/q$. For this study we take $p=1$ and $q=4$, giving four kicks per oscillation period. The twist can then be removed by iterating the mapping four times, keeping only the lowest order terms in $K_a$, to obtain a reduced mapping. Adding random changes in $u$ and $v$ to the reduced mapping we obtain

$$\begin{aligned} v_{n+1}&=v_n-2K_a\sin u_n+u_r\,,\\ u_{n+1}&=u_n+2K_a\sin v_{n+1}+v_r\,, \end{aligned} \tag{1}$$

where $u_r$ and $v_r$ are the extrinsic stochastic components. A piece of the resulting phase space is shown in Fig. 1, for $K_a=0.5$ and $u_r=v_r=0$, for a few initial conditions, showing both the KAM surfaces within a tile, and the stochastic continuous web surrounding the separatrix given to lowest order by

$$v=\pm(u+\pi)+2\pi m\,, \tag{2}$$

where $m$ is an integer. In terms of the normalized action $w$, which goes from zero on the separatrix to unity at the

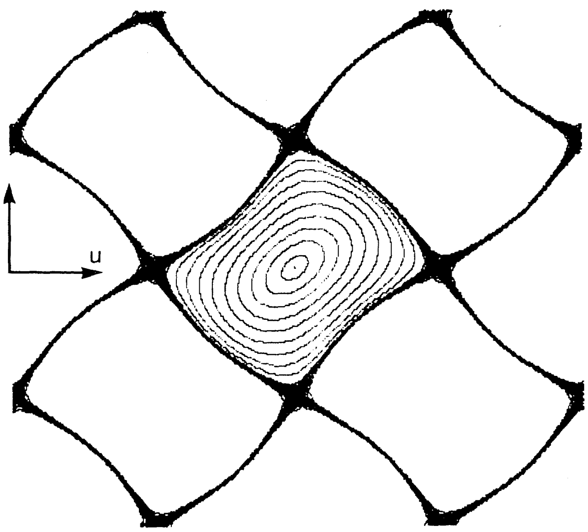


FIG. 1. A portion of the phase space with a 4:1 resonance, showing the stochastic web and a few curves of constant Hamiltonian within a tile.

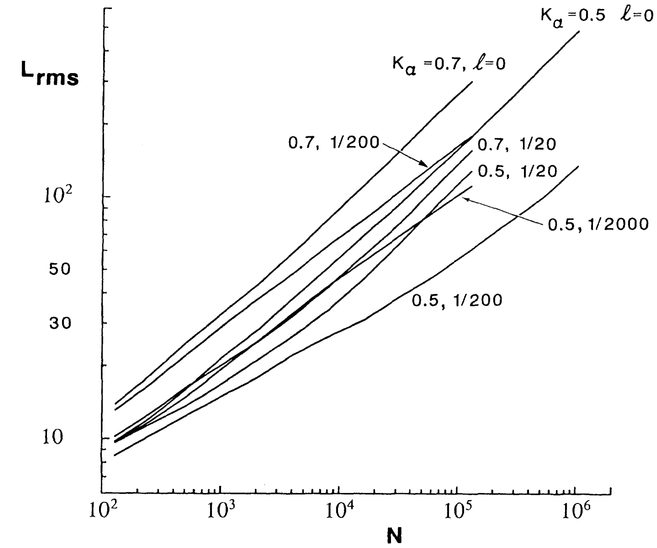


FIG. 2. Diffusive spread of the distribution vs iteration number for a representative sample of values of perturbation parameter $K_\alpha$ and random step $l$. The 1000 initial conditions are within the stochastic web in the neighborhood of an unstable fixed point.

tile center, the following results were obtained, valid near the separatrix, with no extrinsic noise.[7] The rotation period within the tile, in units of the mapping period, is

$$\tau(w) \cong 2(2-\ln w)/K_\alpha . \qquad (3)$$

The thickness of the stochastic web is

$$w_1 = (2\pi/K_\alpha^2)\exp(-\pi^2/2K_\alpha) . \qquad (4)$$

Integrating (3) over $w_1$ we obtain the average $\tau$ over the web

$$\tau_{\mathrm{ave}} = \pi^2/K_\alpha^2 + (6 - 2\ln 20\pi + 6\ln K_\alpha)/K_\alpha . \qquad (5)$$

The number of rotation periods per separatrix crossing is found from the solution of a local diffusion mechanism within the web, assuming nearly uniform local phase space, to be

$$n = \pi^2/4K_\alpha . \qquad (6)$$

We shall see that this local uniformness is also characteristic of the asymptotic diffusion over the entire tile in the presence of noise. Assuming normal random-walk diffusion through the web with a Gaussian distribution on the large-tile space scale

$$W(L) = (\pi N L_{\mathrm{tile}})^{-1}\exp[-L^2/NL_{\mathrm{tile}}] ,$$

then the root-mean-square spreading $L_{\mathrm{rms}}$ is given by

$$L_{\mathrm{rms}} = (D_{\mathrm{web}}N)^{1/2} , \qquad (7)$$

where $N$ is the number of mapping periods, and

$$D_{\mathrm{web}} = L_{\mathrm{tile}}^2/T_{\mathrm{tile}} = 2\pi^2/n\tau_{\mathrm{ave}} , \qquad (8)$$

where $L_{\mathrm{tile}} = \sqrt{2}\,\pi$ is the distance between adjacent tile centers. Equation (8) has been verified, numerically, over a range of $K_\alpha$.[7]

We now introduce extrinsic noise, as a uniformly distributed random variable between $\pm l$, and numerically examine the result. We first examine, directly, the expansion of $L_{\mathrm{rms}}$ with $N$ for a range of $K_\alpha$'s and noise coefficients $l$ in Fig. 2. The steeper slopes of $\frac{1}{2}$, on log-log coordinates, give the usual proportionality for random-walk diffusion $L_{\mathrm{rms}} \propto N^{1/2}$. With no noise, after an initial transient representing spreading within the web away from initial conditions near the unstable fixed point, all $K_\alpha$'s follow this proportionality, with the proportionality constant given by $D_{\mathrm{web}}$ in (8). With a relatively large noise coefficient of $l = \frac{1}{20}$, and for the larger intrinsically stochastic webs, $K_\alpha = 0.5$ and 0.7, after an initial transient period between $10^4$ and $10^5$ iterations with slower than random-walk diffusion, the $L_{\mathrm{rms}} \propto N^{1/2}$ reestablishes itself, but with a lower proportionality constant. For these $K_\alpha$ values, and a smaller noise coefficient of $l = \frac{1}{200}$, the random-walk proportionality has approximately established itself by $N = 10^6$ (only $K_\alpha = 0.5$ is shown over this $N$). For a smaller noise coefficient $l = \frac{1}{2000}$, we obtain a different result in which the diffusion at first follows more closely the random-walk diffusion slope and then falls to a lower proportionality, over the time of observation (illustrated for $K_\alpha = 0.5$ in the figure). To understand these results we show, in Fig. 3, the fraction of phase points in the web $f_N$ versus iteration number, for the values of $K_\alpha$ and noise discussed above. For a noise coefficient of $l = \frac{1}{20}$, after an initial few iterations in which all phase points

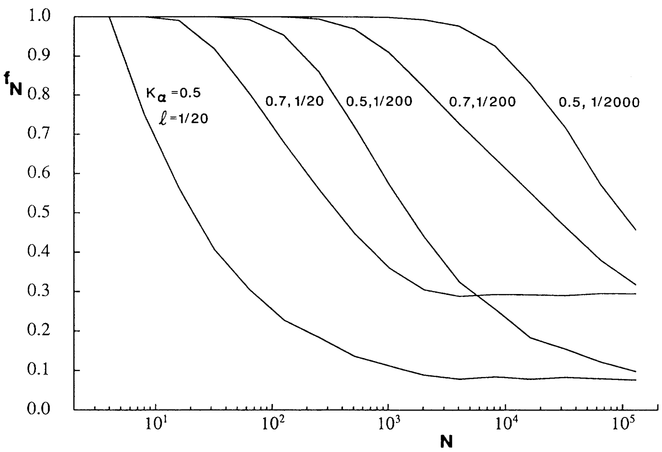


FIG. 3. The fraction of phase points within the web vs iterations, with $K_\alpha$ and $l$ as parameters; 10000 initial conditions in the web.

are in the web, there is a transition to an asymptotic value which occurs for both $K_\alpha=0.5$ and 0.7 at about the same number of iterations between $10^3$ and $10^4$, but with different asymptotes. Referring back to Fig. 2, we see that sometime after the asymptotic ratio of web to tile phase points is reached, the diffusion behaves as a random walk. For the case of $K_\alpha=0.7$, comparing the noise $l=\frac{1}{20}$ case with the no-noise diffusion we find the ratio of iterations for a given $L_{\rm rms}$, $D/D_{\rm web}$, to be the asymptotic $f_N$, found to be 0.3 in Fig. 3. We conclude that the tile particles, in equilibrium with the surrounding web, act as a local source for the web diffusion, as we had previously postulated. A numerical measurement of the area ratio gives $f=0.29$, in good agreement. The diffusion calculation for $K_\alpha=0.5$ with a noise coefficient of $l=\frac{1}{200}$, at an asymptotic value near $N=10^6$, gives $D/D_{\rm web}\cong 0.07$ in good agreement with $f_N\cong 0.07$ found from the asymptotic value in Fig. 3. The measured area ratio is $f=0.065$, again in good agreement.

The transition time to the asymptotic state is related to the extrinsic stochastic filling time for the tile. We use the usual random-walk argument to compute the time to diffuse from the web to the center of the tile, a distance of $\pi/\sqrt{2}$. For noise amplitude $l=\frac{1}{20}$, the average step across KAM surfaces $L_{\rm noise}=1/20\sqrt{2}$, giving a diffusion time $N=(20\pi)^2\simeq 4\times 10^3$. For noise amplitude $l=\frac{1}{200}$ the time is a factor of 100 larger, or $N=4\times 10^5$. These times are consistent with the numerical observations of Fig. 3.

The dynamics, however, include some additional subtleties. The edge of the distribution, i.e., that part created on a time scale shorter than the extrinsic diffusion time across a tile, is clearly not asymptotic. This accounts for the near approach of $f_N$ to the asymptotic value while $D/D_{\rm web}$ has not yet reached its asymptotic value, i.e., while the diffusion rate is not yet $N^{1/2}$.

While we have concentrated our attention on the cases dominated by the large-space-scale (web) diffusion, the web diffusion is not always dominant. Since the random steps at the small space scale take place at each mapping step the extrinsic noise can dominate the diffusion when the noise step is sufficiently large and the time for the intrinsic large-space-scale step is sufficiently slow. In fact, the extrinsic diffusion is itself a two-space-and-time-scale process, when acting upon a phase-space topology of the type considered here (e.g., Fig. 1). Local diffusion within each tile results in separatrix crossings. Because of the rotation of particles within each tile on a fast time scale the result is to step the crossing particles effectively the distance between tile centers. The global extrinsic diffusion, $D_{\rm ex}$, is then governed by the step size of a tile and a characteristic time for the exchange of tile phase space across the bounding separatrices,

$$D_{\rm ex}=L_{\rm tile}^2/\tau_{\rm ex}\,; \tag{9}$$

we can characterize $\tau_{\rm ex}$ by

$$\tau_{\rm ex}=(f_{\rm ex}P_{\rm ex})^{-1}\,, \tag{10}$$

where $f_{\rm ex}$ is the fraction of phase space that is accessible to crossing the separatrix on each mapping step due to the extrinsic stochasticity, and $P_{\rm ex}$ is the probability for that phase space to actually cross. We use simple geometric arguments to calculate these quantities. As mentioned earlier, for our numerical investigation we have chosen $u_r$ and $v_r$ to be uniformly distributed random variables between $\pm l$. The accessible phase space consists of four edge strips of length $\pi\sqrt{2}$ and width $l\sqrt{2}$, while the entire tile phase space is $2\pi^2$. The fraction of accessible phase space is then

$$f_{\rm ex}=4l/\pi\,. \tag{11}$$

For uniformly distributed $u_r$ and $v_r$, the probability of a step a distance $\delta$ across curves of constant Hamiltonian can be directly calculated to be

$$g(\delta)=(1/2l^2)(\sqrt{2}l-\delta)\,.$$

The phase-space crossing probability in the accessible layer is then

$$P_{\rm ex}=\frac{1}{\sqrt{2}l}\int_0^{\sqrt{2}l}dx\int_x^{\sqrt{2}l}g(\delta)d\delta=\tfrac{1}{12}\,. \tag{12}$$

Substituting (11) and (12) in (10) results in

$$\tau_{\rm ex}=3\pi/l\,. \tag{13}$$

Extrinsic diffusion dominates the intrinsic diffusion for $\tau_{\rm ex}<n\tau_{\rm ave}/f$, with the rate of diffusion then characterized by (9).

The competition is illustrated numerically in Fig. 4, in which for a fixed $N$, $L_{\rm rms}$ is plotted against $K_\alpha$, with noise as a parameter. The case with no noise has the expected dependency $L_{\rm rms}\propto K_\alpha^{3/2}$. The diffusion with extrinsic

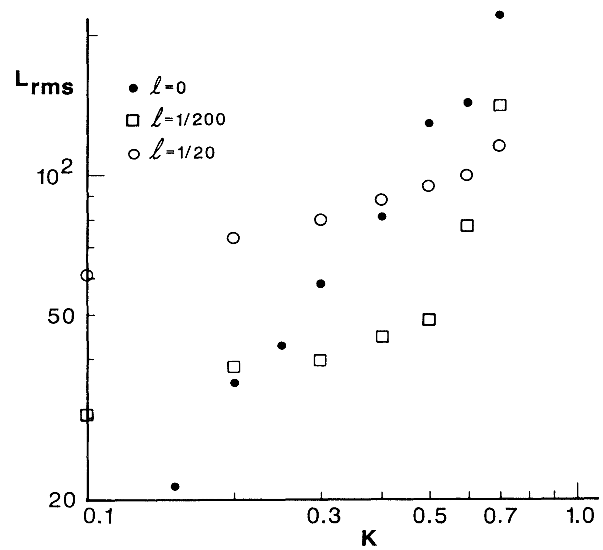


FIG. 4. Diffusive spread of the distribution vs $K_a$ with $l$ as a parameter, after $2^{16}$ iterations for 1000 initial conditions in the web.

noise lies above the diffusion without noise for values of $K_a$ at which the extrinsic diffusion is larger than the web diffusion. For $K_a=0.1$, we expect the diffusion to be governed by the extrinsic noise alone. For noise $l=\frac{1}{20}$, using (9) and (13) we calculate after $2^{16}$ iterations that $L_{\rm rms}=82$. This is significantly higher than the value $L_{\rm rms}=60$, found numerically, which indicates a hidden missing factor that increases $\tau_{\rm ex}$ over the simple analytic estimate. One possibility is a nonuniformity in the phase that may develop if the rotation frequency near the separatrix [see Eq. (3)] is too slow. The gentle upward slant of $L_{\rm rms}$ as a function of $K_a$ in the cases dominated by extrinsic stochasticity may be indicative of this. The lower-noise case $l=\frac{1}{200}$ does not lie a factor of $10^{-1/2}$ below the $l=\frac{1}{20}$ case at $K_a=0.1$ because after $2^{16}$ iterations it is not yet asymptotic; the occupied phase space within the tile regions is therefore smaller. The other interesting feature of Fig. 4 is the break in the slopes of $L_{\rm rms}$ with noise. These should occur when $\tau_{\rm tile}\approx\tau_{\rm ex}$, i.e., when

$$\frac{\pi^4}{4K_a^3 f(K_a)}=\frac{3\pi}{l}, \tag{14}$$

which we have verified, numerically, by the positions of the slope changes in Fig. 4.

In conclusion, we are able to understand two-time-step diffusion in terms of establishing a local quasiequilibrium. The technique can be used to calculate the diffusion through a divided phase space consisting of both intrinsic and extrinsic diffusion, provided the structure of the phase space is sufficiently uniform on the length scale $L_{\rm rms}$ over which the diffusion is to be calculated.

This work was supported in part by the Office of Naval Research Grant No. N00014-89-J-1097.

[1]B. V. Chirikov, Phys. Rep. **52**, 265 (1987).

[2]A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, Berlin, 1983).

[3]R. W. White, in *Statistical Physics and Chaos in Fusion Plasma,* edited by G. W. Horton and L. E. Reichl (Wiley, New York, 1984), pp. 209–224.

[4]A. B. Rechester, M. N. Rosenbluth, and R. B. White, Phys. Rev. A **23**, 2664 (1981).

[5]C. F. F. Karney, Phys. Fluids **21**, 1584 (1978); **22**, 2188 (1979).

[6]G. M. Zaslavskii, M. Yu. Zakharov, R. Z. Sagdeev, D. A. Usikov, and A. A. Chernikov, Zh. Eksp. Teor. Fiz. **91**, 500 (1986) [Soviet Phys. JETP **64**, 294 (1986)]; Pis'ma Zh. Eksp. Teor. Fiz. **44**, 349 (1986) [JETP Lett. **44**, 453 (1986)].

[7]A. J. Lichtenberg, and B. P. Wood, Phys. Rev. **39**, 2153 (1989).