Scaling Exponents in Nonisotropic Convective Turbulence

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A dynamical theory for the structure functions in convective turbulence at high Rayleigh numbers is presented. A range of scales is identified in which buoyancy forces dominate and the scaling exponents differ significantly from isotropic turbulence. A crossover to Kolmogorov-type scaling is predicted as a function of the Rayleigh number.

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This Letter is motivated by recent experiments[1] on turbulent motion in convective cells at very high Rayleigh numbers (currently as high as $R \sim 10^{14}$). There are preliminary indications that in this type of turbulence there exists scaling behavior, but the exponents characterizing the scaling of correlations functions are not the usual ones obtained for isotropic turbulence.[2]

Denoting by $\mathbf{u}$ and T the velocity and temperature fields, respectively, one considers the r dependence of correlation functions such as

$$\langle\langle u_i(\mathbf{x}+\mathbf{r})u_i(\mathbf{x})\rangle\rangle \sim r^\alpha , \tag{1a}$$

$$\langle\langle T(\mathbf{x}+\mathbf{r})T(\mathbf{x})\rangle\rangle \sim r^\beta , \tag{1b}$$

$$\langle\langle T(\mathbf{x}+\mathbf{r})u_i(\mathbf{x})\rangle\rangle \sim r^\xi , \tag{1c}$$

where the double angular brackets denote ensemble averages. For isotropic turbulence the Kolmogorov approach[3] predicts $\alpha=\beta=\frac{2}{3}$. The correlation between T and u_i vanishes in isotropic systems, whereas in a Rayleigh-Benard configuration with a temperature gradient in the $i=3$ direction we expect $\langle\langle T(\mathbf{x}+\mathbf{r})u_3(\mathbf{x})\rangle\rangle$ to exist. On general grounds all that can be said is that the Cauchy-Schwartz inequality guarantees that

$$\langle\langle T(\mathbf{x}+\mathbf{r})u_3(\mathbf{x})\rangle\rangle \le [\langle\langle T(\mathbf{x}+\mathbf{r})T(\mathbf{x})\rangle\rangle\langle\langle u_3(\mathbf{x}+\mathbf{r})u_3(\mathbf{x})\rangle\rangle]^{1/2} ,$$

although from dimensional considerations we expect

$$2\xi = \alpha+\beta . \tag{2}$$

In this Letter we report a dynamical theory that shows that buoyancy forces are responsible for a scale-dependent driving of turbulence causing a fundamental departure from the Kolmogorov picture.[3] There are two ranges of scales; denoting by L_0 and l_d the size of the system and the dissipative scale, respectively,[3] one identifies[4,5] a length l_b above which buoyancy effects are dominant. For $l_b \ll l \ll L_0$ we find different scaling exponents from the range $l_d \ll l \ll l_b$ where standard exponents are expected. The length l_b is shown here to depend on the Rayleigh number R, and is predicted to scale like $L_0 R^{-3/28}$.

In developing our theory we make use of a recent formalism advanced by Effinger and Grossmann for isotropic turbulence.[6] At the heart of this approach lies a separation of large-scale from small-scale motions.[7] Considering a field $A(\mathbf{x},t)$ one defines a running average $A^{(r)}(\mathbf{x},t)$ by

$$A^{(r)}(\mathbf{x},t) = \frac{3}{4\pi r^3}\int_{|y|\le r} d\mathbf{y}\, A(\mathbf{x}+\mathbf{y},t) \tag{3}$$

and a complement

$$\tilde{A}^{(r)}(\mathbf{x},t) \equiv A(\mathbf{x},t) - A^{(r)}(\mathbf{x},t) .$$

Physically one expects $A^{(r)}$ to capture long-range motions whereas $\tilde{A}^{(r)}$ reflects locally short-range motions. The parameter r is left unspecified and carries the scaling information. It is easy to show that in homogeneous (but not necessarily isotropic) systems if $\langle\langle |A(\mathbf{x}+\mathbf{r},t)-A(\mathbf{x},t)|^2\rangle\rangle \sim r^\nu$ also $\langle\langle A^{(r)}(\mathbf{x},t) \times A^{(r)}(\mathbf{x},t)\rangle\rangle \sim r^\nu$. The scaling exponent of the former is experimentally accessible, but theoretically we calculate it via the latter.

The theoretical analysis of convective turbulence is based on the Boussinesq approximation[1,8]

$$\partial u_i(\mathbf{x},t)/\partial t = -u_j(\mathbf{x},t)\partial_{x_j}u_i(\mathbf{x},t) - \partial_{x_i}p(\mathbf{x},t) + \nu\nabla^2 u_i(\mathbf{x},t) - (1-\alpha T)g\delta_{i3} , \tag{4}$$

$$\partial T(\mathbf{x},t)/\partial t = -u_j(\mathbf{x},t)\partial_{x_j}T(\mathbf{x},t) + \kappa\nabla^2 T(\mathbf{x},t) + f(\mathbf{x},t) . \tag{5}$$

Here p, ν, α, g, and κ are the pressure, kinematic viscosity, volume expansion coefficient, gravitational acceleration, and heat diffusivity, respectively. It is assumed that the fluid is in a box heated from below and $f(\mathbf{x},t)$ represents the effect of the boundary conditions on the temperature. $f(\mathbf{x},t)$ will be taken as containing large-scale components only. Notice that in isotropic turbulence one assumes such a forcing on the velocity.[6] Here the forcing on the velocity is solely due to buoyancy as displayed in Eq. (4). An auxiliary equation is the incompressibility condition $\partial_{x_i}u_i = 0$.

For convenience we shall denote the smoothing average of Eq. (3) by a single bracket, i.e., $A^{(r)}(\mathbf{x},t) \equiv \langle A(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)}$.

An approximation that will be used below is that $\langle A^{(r)}(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)} = A^{(r)}(\mathbf{x},t)$, and consequently $\langle \tilde{A}^{(r)}(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)} = 0$. For a comprehensive discussion of this approximation see Ref. 6. Using this approximation Eqs. (4) and (5) turn into the following equations for the superscale fields:

$$\partial_t u_i^{(r)} = -u_j^{(r)}\partial_{x_j} u_i^{(r)} - \langle \tilde{u}_j^{(r)}(\mathbf{x}+\mathbf{y},t)\partial_{x_j}\tilde{u}_i^{(r)}(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)} - \partial_{x_i} p^{(r)} + \nu\nabla^2 u_i^{(r)} - (1-\alpha T^{(r)})g\delta_{i3}, \tag{6a}$$

$$\partial_t T^{(r)} = -u_j^{(r)}\partial_{x_j} T^{(r)} - \langle \tilde{u}_j^{(r)}(\mathbf{x}+\mathbf{y},t)\partial_{x_j}\tilde{T}^{(r)}(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)} + \kappa\nabla^2 T^{(r)} + f^{(r)}, \tag{6b}$$

and the following for the subscale fields:

$$(\partial_t + u_j\partial_{x_j})\tilde{u}_i^{(r)} = -\tilde{u}_j\partial_{x_j} u_i^{(r)} - \partial_{x_j}\tilde{p}^{(r)} + \nu\nabla^2\tilde{u}_i^{(r)} + \langle \tilde{u}_j(\mathbf{x}+\mathbf{y},t)\partial_{x_j}\tilde{u}_i(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)} + \alpha g\tilde{T}^{(r)}\delta_{i3}, \tag{7a}$$

$$(\partial_t + u_j\partial_{x_j})\tilde{T}^{(r)} = -\tilde{u}_j\partial_{x_j} T^{(r)} + \kappa\nabla^2\tilde{T}^{(r)} + \langle \tilde{u}_j(\mathbf{x}+\mathbf{y},t)\partial_{x_j}\tilde{T}^{(r)}(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)}. \tag{7b}$$

These equations are valid generally (within the stated approximations) for any Boussinesqian system. The system at hand has the following simplifying features: Experimentally it has been found[1] that most of the average temperature gradient is confined to thin boundary layers whereas the core of the fluid is almost homogeneous. It is natural therefore to idealize the situation to a model of stationary, homogeneous, but nonisotropic fluid.[8] Using these properties we can turn Eqs. (6) into balance equations for the correlations of superscale fields by multiplying Eq. (6a) by $u_i^{(r)}$ and Eq. (6b) by $T^{(r)}$, and ensemble averaging. Remembering the incompressibility conditions, the resulting equations are

$$\langle\langle u_i^{(r)}\langle \tilde{u}_j^{(r)}(\mathbf{x}+\mathbf{y},t)\partial_{x_j}\tilde{u}_i^{(r)}(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)}\rangle\rangle - \nu\langle\langle u_i^{(r)}\nabla^2 u_i^{(r)}\rangle\rangle = \alpha g\langle\langle u_3^{(r)} T^{(r)}\rangle\rangle, \tag{8a}$$

$$\langle\langle T^{(r)}\langle \tilde{u}_j^{(r)}(\mathbf{x}+\mathbf{y},t)\partial_{x_j}\tilde{T}^{(r)}(\mathbf{x}+\mathbf{y},t)\rangle_y^{(r)}\rangle\rangle - \kappa\langle\langle T^{(r)}\nabla^2 T^{(r)}\rangle\rangle = \langle\langle T^{(r)} f^{(r)}\rangle\rangle. \tag{8b}$$

Notice that in the standard case of isotropic turbulence[6] the right-hand side (RHS) of Eq. (8a) is a forcing term of the type $\langle\langle u^{(r)} f_u^{(r)}\rangle\rangle$. Assuming that f_u operates on large scales only, this term becomes a constant for $r \ll L_0$, and is identified as the energy injection rate ϵ. In the present case $\langle\langle T^{(r)} f^{(r)}\rangle\rangle$ becomes constant but the forcing on u is r dependent on all scales. Physically this stems from the fact that eddies of different sizes gain different amounts of energy from the gravity field. This r-dependent forcing will be responsible for the departure from the Kolmogorov scaling. The terms on the left-hand side (LHS) of Eqs. (8) have the interpretation of energy transfer and viscous (or thermal) dissipation, respectively. The first terms in both equations will be shown to be related to eddy viscosity and eddy conductivity, respectively.

To proceed we have to treat the first terms on the LHS of Eqs. (8) and convert them into forms containing two-point correlations. The idea is to solve for $\tilde{u}_i^{(r)}(\mathbf{x}+\mathbf{y},t)$ and $\tilde{T}^{(r)}(\mathbf{x}+\mathbf{y},t)$ by integrating Eqs. (7) along a Lagrangian path. Both equations have the form

$$(\partial_t + u_j\,\partial_{x_j})\tilde{A}^{(r)}(\mathbf{x},t) = \mathcal{F}_{\mathrm{RHS}}(\mathbf{x},t).$$

Accordingly a formal solution is furnished by

$$A(\mathbf{z},t) = \int_{-\infty}^{t} dt'\,\mathcal{F}_{\mathrm{RHS}}(x(t';z,t),t) + \tilde{A}(r)\quad (t=-\infty), \tag{9}$$

where $x(t';z,t)$ is the path taken by a fluid particle that reaches $\mathbf{x}=\mathbf{z}$ when $t'=t$. Substituting such a formal solution into the relevant terms in Eqs. (8) we find that under the stated approximations many terms vanish. Those that remain are all given as four-point correlation functions containing two superscale fields and two subscale fields. These shall be approximated as products of two-point correlations, one in the subscale and the other in the superscale fields. Notice that this is *not* a Gaussian approximation but a dynamical statement about decoupling of fluctuations occurring on widely different length scales. Nevertheless we stress that any exponent found below should be understood as a mean-field value.

Performing these operations and using the expression for the pressure,

$$\partial_{x_i} p = \int d\mathbf{x}'\,G(\mathbf{x}')\partial_{x_i}[\partial_{x_l} u_k(\mathbf{x}+\mathbf{x}',t)\partial_{x_k} u_l(\mathbf{x}+\mathbf{x}',t)],$$

where G is the Green's function that solves $\nabla^2 G(x,x') = \delta(x-x')$, we end up with the equations

$$\begin{aligned} &-\tfrac{1}{2} N_{u_k u_l}^{(r)}(0)\partial_{x_k'}\partial_{x_l'} R_{u_i u_i}^{(r)}(\mathbf{x}')\big|_{x'=0} \\ &\quad + \int d\mathbf{x}'\,G(\mathbf{x}')\partial_{x_i}\partial_{x_j'}[N_{u_k u_l}^{(r)}(\mathbf{x}')\partial_{x_l'}\partial_{x_m'} R_{u_i u_j}(\mathbf{x}') + N_{u_k u_l}^{(r)}(\mathbf{x}')\partial_{x_l'}\partial_{x_i'} R_{u_l u_i}(\mathbf{x}')] \\ &\qquad - \nu\langle\langle u_i^{(r)}(\mathbf{x},t)\nabla^2 u_i^{(r)}(\mathbf{x},t)\rangle\rangle = \alpha g\langle\langle u_3^{(r)}(\mathbf{x},t) T^{(r)}(\mathbf{x},t)\rangle\rangle, \end{aligned} \tag{10a}$$

and

$$\begin{aligned} &-\tfrac{1}{2} N_{u_k u_l}^{(r)}(0)\partial_{x_k'}\partial_{x_l'} R^{(r)}(\mathbf{x}')\big|_{x'=0} + \int d\mathbf{x}'\,G(\mathbf{x}')\partial_{x_i'}\partial_{x_j'}[N_{T u_l}^{(r)}(\mathbf{x}')\partial_{x_l'}\partial_{x_l'} R_{T u_j}^{(r)}(\mathbf{x}')] \\ &\qquad - \kappa\langle\langle T^{(r)}(\mathbf{x},t)\nabla^2 T^{(r)}(\mathbf{x},t)\rangle\rangle = \langle\langle T^{(r)}(\mathbf{x},t) f^{(r)}\rangle\rangle, \end{aligned} \tag{10b}$$

where

$$R_{\alpha\beta}^{(r)}(\mathbf{r}') = \langle\langle\langle \alpha^{(r)}(\mathbf{x}+\mathbf{y},t)\beta^{(r)}(\mathbf{x}+\mathbf{y}+\mathbf{r}',t)\rangle\rangle\rangle_y^{(r)}, \tag{11a}$$

$$N_{\alpha\beta}^{(r)}(\mathbf{r}') \equiv \int_{-\infty}^{t} \langle\langle \tilde{\alpha}^{(r)}(z,t)\tilde{\beta}^{(r)}(\mathbf{x}(t';z,t)+\mathbf{r}',t')\rangle\rangle dt'. \tag{11b}$$

Notice that $N_{\alpha\beta}$ of Eq. (11b) has the form of a Green-Kubo transport coefficient where molecular fluxes are replaced by small-scale field components. These dress the molecular coefficients ν and κ.

The last crucial step of analysis involves treating the time correlation functions in $N_{\alpha\beta}^{(r)}$. We follow the spirit of Ref. 6, except that in this case the analysis is significantly more complex due to the anisotropy and the coupling between the velocity and temperature fields. The analysis results in the relations

$$N_{\alpha\beta}^{(r)}(\mathbf{r}') = \tilde{C}_{\alpha\gamma}^{(r)}(\mathbf{r}')(\tilde{\Gamma}^{(r)})_{\gamma\delta}^{-1}\tilde{C}_{\delta\beta}^{(r)}(\mathbf{r}'), \tag{12}$$

where

$$\tilde{C}_{\alpha\beta}^{(r)}(\mathbf{r}') = \langle\langle \tilde{\alpha}^{(r)}(\mathbf{x},t)\tilde{\beta}^{(r)}(\mathbf{x}+\mathbf{r}',t)\rangle\rangle, \tag{13a}$$

$$\tilde{\Gamma}_{\alpha\beta}^{(r)}(\mathbf{r}') = -\langle\langle \tilde{\alpha}^{(r)}(\mathbf{z},t)d_{t'}\tilde{\beta}(x(t';\mathbf{z},t)+\mathbf{r}',t')\rangle\rangle\big|_{t=t'}. \tag{13b}$$

Using these results Eqs. (10) can be turned into coupled integro-differential equations in the relevant structure functions. For the purposes of this Letter we stress the scaling laws that are implied by Eqs. (10). Using the three exponents α, β, and ζ introduced in Eq. (1), i.e., $\langle\langle u_3^{(r)}(\mathbf{x},t)u_3^{(r)}(\mathbf{x},t)\rangle\rangle \sim r^{\alpha}$, $\langle\langle T^{(r)}(\mathbf{x},t)T^{(r)}(\mathbf{x},t)\rangle\rangle \sim r^{\beta}$, and $\langle\langle u_3^{(r)}(\mathbf{x},t)T^{(r)}(\mathbf{x},t)\rangle\rangle \sim r^{\zeta}$, we find from Eqs. (10)–(13), after some algebra, in the limit of large r,

$$r^{\max(2\alpha,\alpha+\beta+\zeta,3\zeta)-\max(\zeta,2\beta)+\alpha-2} + r^{\alpha-2} \sim r^{\zeta}, \tag{14a}$$

$$r^{\max(2\alpha,\alpha+\beta+\zeta,3\zeta)-\max(\zeta,2\beta)+\beta-2} + r^{\zeta-\max(\zeta,2\beta)+\max(\zeta+\alpha,\beta+2\zeta,2\beta+\alpha,2\beta+\zeta)-2} + r^{\beta-2} \sim r^{0}. \tag{14b}$$

Notice that in the isotropic case[6] Eq. (14a) trivializes to

$$r^{3\alpha-2} + r^{\alpha-2} \sim r^{0}, \tag{15}$$

where for r large and $\alpha<2$ implies the celebrated $\alpha=\frac{2}{3}$ result of Kolmogorov. Examining Eqs. (14) we see immediately that a solution

$$\alpha=\tfrac{6}{5},\quad \beta=\tfrac{2}{5},\quad \zeta=\tfrac{4}{5}, \tag{16}$$

trivializes all the max functions and satisfies all the exponent equalities. Using also the condition (2) it is found that Eq. (16) is the unique solution of Eqs. (14).

It is amusing to notice that the same exponents have been suggested in Refs. 4 and 5 for turbulence in a *stably* stratified medium, in which gravity bleeds energy away from the turbulence. Our case is opposite, since buoyancy drives our turbulence. Apparently the dimensional arguments in Refs. 4 and 5, which appear somewhat *ad hoc*, capture the essential physics which is spelled out in our dynamical calculation.

Finally we need to estimate the length scale l_b below which a crossover to Kolmogorov exponents is expected.[4,5] The parameters that determine this length are αg of Eq. (4) and the mean heat flux $J=\langle\langle u_3(\mathbf{x})T(\mathbf{x})\rangle\rangle$, on the one hand, and the rate of energy transfer $\epsilon \sim u_0^3/L_0$, on the other hand, where u_0 is the typical large-scale (L_0) velocity in the core of the fluid. We expect that the nonlinearities would tend to isotropize the flow, such that below some length scale isotropic models should hold. This length can be formed from the parameters ϵ, $a_1 = J(\partial T/\partial z)$, and αg by

$$l_b = \epsilon^{5/4}/a_1^{3/4}(\alpha g)^{3/2}. \tag{17}$$

To estimate l_b up to prefactors ~ 1, we use the scaling laws of Ref. 1 in terms of the Rayleigh number $R = \alpha g \Delta L_0^3/\kappa\nu$, where Δ is the temperature difference on the boundaries of the box. Reference 1 offers scaling laws for the Nusselt number Nu and the typical velocity u_0:

$$\mathrm{Nu} \sim R^{2/7}, \tag{18a}$$

$$u_0 \sim R^{3/7}\nu/L_0. \tag{18b}$$

In these scaling laws there should also be a Prandtl number dependence which we suppress since it is not known experimentally to any acceptable precision.[9] Using these results in (17) we derive

$$l_b \sim R^{-3/28}L_0, \tag{19}$$

up to numbers ~ 1 and a weak dependence on the Prandtl number. Thus for R of the order of 10^{14} we expect about a decade of length scales over which the exponents (16) are measurable. For higher Rayleigh numbers this range of scales should increase, as the crossover to Kolmogorov scaling is pushed further down to smaller scales.

In summary, we have demonstrated that a dynamical calculation yields scaling laws for the relevant correlation functions in unstably stratified turbulence that are in marked difference from Kolmogorov scaling, but in agreement with dimensional arguments for the scaling laws in stably stratified turbulence. We hope that the experimental investigations would rapidly reach a level where detailed comparisons with this theory are possible.

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