Quantum Optical Fredkin Gate

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A simple optical model to realize a reversible, potentially error-free logic gate—a Fredkin gate—is discussed. The device dissipates no energy and makes use of the Kerr nonlinearity to produce intensity-dependent phase shifts. The analysis shows that quantum mechanics permits the operation of error-free logic gates which dissipate no energy. However, even though the device is nondissipative, error-free performance only occurs under particular operating conditions.

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Computers are constructed from physical devices and as such are constrained by the laws of physics. Do these laws place universal limitations on computation?[1] Apparently not, providing the computer is constructed from so-called "reversible logic gates."[2-4] All current computers are open dissipative systems requiring an energy input to run. The physical elements which realize the logical primitives of the computer are dissipative. However, it is not necessary that logical primitives be realized by dissipative elements. An example will be given in this Letter. A device which dissipates no energy is potentially reversible. Thus we are led to consider reversible logic gates. Such a device, however, need not be error free; that is, its output may have been "1" when it should have been "0."

Does quantum mechanics impose any fundamental limits to computation, even for those performed by reversible gates? The consensus appears to be that it does not.[4-10] This conclusion is based on analyzing particular, rather idealized models for reversible computation, according to quantum-mechanical principles. Benioff[5] discussed a quantum model for a standard Turing machine. Feynman[4] has proposed a model for a reversible logic gate based on a two-state quantum system, such as spin. A somewhat more realistic, though less ideal model based on an ac SQUID has been proposed by Likharev,[11] while Obermayer, Mahler, and Haken[12] have suggested a solid-state bistable device. These devices, however, are not isolated from thermal or quantum fluctuations in the environment. An optical Fredkin gate has been proposed by Shamir *et al.*[13]

In this paper a reversible logic gate constructed from a Mach-Zehnder interferometer and a crystal with an intensity-dependent refractive index will be discussed. The device may be operated at the "quantum level" with single photons carrying the logical status, or at a macroscopic level with light pulses. The device is all optical. By operating at optical frequencies it is essentially isolated from thermal noise and is thus the ideal device for analyzing the ultimate quantum limits to reversible computation. The conclusions can be described as follows. A device realizing a logical primitive has a certain number of inputs and outputs. If it is dissipative, energy is lost from some or all of the inputs and noise is necessarily added to the corresponding outputs. Even at absolute zero, dissipation on any input line leads quantum (zero-point) noise to be added to the output.[14] This is necessary to preserve the commutation relations for the operators describing the output states and is essentially a consequence of the fluctuation-dissipation theorem. If the device is nondissipative, noise may still be added to the outputs (fluctuations are not necessarily accompanied by dissipation). However, it is always possible to conceive of a nondissipative device which adds no noise to the output. Of course, energy may be dissipated in reading the final output; however, this depends only on the number of output lines and not on the number of computation steps.

One particular logical primitive for reversible computation has been described by Fredkin and Toffoli[3] and will be referred to here as a Fredkin gate. This is a device with three input lines and three output lines. One of the lines is designated the control line and the logical status of this line is left unchanged by the gate. If the bit on the control line is zero, the logical status of the other two lines is unchanged. If the bit on the control line is one, the bits on the other two lines are interchanged. The logic diagram for the gate is shown in Table I. The reader is referred to Ref. 3 for details on how AND gates, flip-flops, etc., may be constructed from Fredkin gates.

An optical model for a Fredkin gate is indicated

TABLE I. Logic table for a Fredkin gate.

Input			Output		
c_i	a_i	b_i	c_o	b_o	a_o
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1

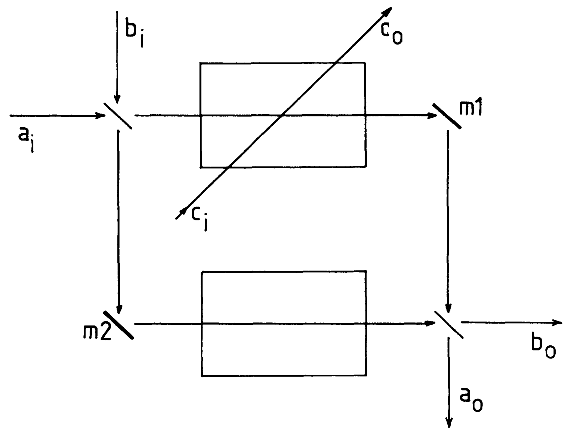

FIG. 1. Schematic outline of an optical system to realize a Fredkin gate. See text for details.

schematically in Fig. 1. Essentially it is a Mach-Zehnder interferometer. A substance with an intensity-dependent refractive index (optical Kerr effect[15]) is placed in both arms. In such a medium the field encounters a refractive index which changes according to the field intensity and thus undergoes an intensity-dependent phase shift. The possibility of using this effect as an optical switch was demonstrated many years ago.[16] It will be assumed that this effect can be adequately described quantum mechanically in terms of a phenomenological third-order nonlinear susceptibility, $\chi^{(3)}$, with quantized fields. This approach has been successfully applied to a number of experiments such as squeezed-state generation[17] and quantum nondemolition detection[18,19] in which quantum effects are exhibited. The appropriate interaction Hamiltonian for the two-field Kerr effect is given in Ref. 17. The device couples three traveling-wave modes of the electromagnetic field. Two input fields represented by annihilation operators a_i and b_i are coupled by the input beam splitter. A third field, represented by the annihilation operator c_i, is the control field and is coupled to the field in one arm of the interferometer, by the nonlinear substance. This field does not pass through the interferometer. In the absence of a control field, the field in each arm of the interferometer undergoes a self-induced intensity-dependent phase shift. When the control field is present, however, it causes an induced phase shift in that arm of the interferometer. It is this phase shift which enables the field c_i to control the output state of the device. The annihilation operators representing the output fields are denoted a_o, b_o, and c_o.

For fields at optical frequencies, the mean thermal proton number in the input fields may be taken as zero. If we further stipulate that mirrors M1 and M2 (see Fig. 1) are lossless and that the nonlinear substance itself is lossless and contributes no spontaneously emitted photons, the three traveling-wave modes are completely decoupled from fluctuations (zero-point or thermal) in any other field mode. These assumption will be reexamined towards the end of this Letter.

The device works as follows. The interferometer is adjusted so that there is maximum transmission in the output mode a_o, when a nonzero field is present at a_i and vacuum states for the fields b_i and c_i. If a_i happens to be in a one-photon state, a photon will be transmitted with certainty in mode a_o. If the input control field contains a field just sufficient to cause a phase change of π in one arm of the interferometer, the intensity in mode a_o falls to zero and the photon is never transmitted in that mode. In a macroscopic version of the gate it is not necessary to operate between the absolute minimum and absolute maximum of the interference pattern, so long as the change in intensity is resolvable above the noise level. However, as will be shown, the single-photon version of the device performs without error, only if operated between 0 and 1 in the interference pattern.

The quantum analysis of the device consists in finding the unitary (nonlinear) transformation connecting the operators a_i, b_i, and c_i to a_o, b_o, and c_o. If we assume 50/50 beam splitters, the required transformation is found to be

$$a_o = \hat{U} a_i - \hat{V} b_i \,, \tag{1}$$

$$b_o = \hat{V} a_i + \hat{U} b_i \,, \tag{2}$$

$$c_o = \exp[(i/2)\chi c_i^\dagger c_i + i\chi a_1^\dagger a_1] c_i \,, \tag{3}$$

where

$$\hat{U} = \tfrac{1}{2}\{\exp[i(\chi/2) a_2^\dagger a_2] + \exp[i\chi(c_i^\dagger c_i + \tfrac{1}{2} a_1^\dagger a_1) + i\theta]\} \,, \tag{4}$$

$$\hat{V} = \tfrac{1}{2}\{\exp[i(\chi/2) a_2^\dagger a_2] - \exp[i\chi(c_i^\dagger c_i + \tfrac{1}{2} a_1^\dagger a_1) + i\theta]\} \,, \tag{5}$$

with

$$a_1 = (1/\sqrt{2})(a_i + b_i), \quad a_2 = (1/\sqrt{2})(a_i - b_i) \,.$$

The operators a_1 and a_2 describe the fields in arms 1 and 2, respectively. The phase θ accounts for any linear phase shift between the two arms of the interferometer and χ is the nonlinear coupling constant proportional to $\chi^{(3)}$.[17] These transformations are derived from the Hamiltonian discussed in Ref. 17. It is easily verified that the commutation relations for the output fields are the same as those of the input.

The mean and variance of the photon number in each of the three output modes may now be determined for various input states. As an example, consider first the case where mode a_i is in a one-photon state and all other inputs are in the vacuum state. The mean and variances in each of the output fields are

$$\langle a_o^\dagger a_o \rangle = 1 - \langle b_o^\dagger b_o \rangle = \tfrac{1}{2}(1 + \cos\theta) \,, \tag{6}$$

$$\langle c_o^\dagger c_o \rangle = \langle c_i^\dagger c_i \rangle = 0 \,, \tag{7}$$

$$V(a_o^\dagger a_o) = V(b_o^\dagger b_o) = \tfrac{1}{4}\sin^2\theta \,, \tag{8}$$

$$V(c_o^\dagger c_o) = 0 \,. \tag{9}$$

If the device is operated at $\theta=2n\pi$ (i.e., at the maximum transmission in mode a_o), the photon is transmitted with certainty in mode a_o [as indicated by $V(a_o^\dagger a_o)=0$ at this point]. If both modes a_i and c_i are in one-photon states then

$$\langle a_o^\dagger a_o\rangle = 1-\langle b_o^\dagger b_o\rangle = \tfrac{1}{2}[1+\cos(\chi+\theta)]\,, \tag{10}$$

$$\langle c_o^\dagger c_o\rangle = \langle c_i^\dagger c_i\rangle = 1\,, \tag{11}$$

$$V(a_o^\dagger a_o) = V(b_o^\dagger b_o) = \tfrac{1}{4}\sin^2(\chi+\theta)\,, \tag{12}$$

$$V(c_o^\dagger c_o) = 0\,. \tag{13}$$

If it can be arranged that $\chi+\theta=\pi$, the photon is never transmitted in mode a_o; that is, it is transmitted with certainty in mode b_o. Continuing in this fashion, one verifies that the device realizes a Fredkin gate with a photon carrying the logical status "1" and that furthermore the device operates without error under these conditions.

Needless to say, it would be extremely difficult to make the device operate at the one-photon level. Huge third-order susceptibilities would be required for the stated phase shifts. The point of the model, however, is that it can in principle be operated in this way. However, some words of caution must be added. The unitary transformations which describe the operation of the device make no reference to the quantum state of the various beams. In this Letter these states are taken to be n-photon pure states. This is at best only a crude idealization of the sort of states appropriate to digital signals. A more complete description would need to consider quantum pulse states, in which case the unitary transformation method could be generalized along the lines of Ref. 20. This extension is currently under way. One may also question the validity of a description in terms of a nonlinear polarizability with beams so weak. The known conditions for the validity of such a description specify that the fields be detuned from a resonance and sufficiently far from saturation, but make no mention of how low the field intensity may be made. It must be admitted that there is some uncertainty in this aspect. Despite these misgivings the model of this paper is offered as a reasonably simple way to explore in more detail possible general quantum limits to computation and as a basis for further work.

It would appear at first sight that the device would operate at the macroscopic level for any sort of input fields. This is not the case. Any intensity fluctuations, quantum or classical, on the control beam will cause phase fluctuations in one arm of the interferometer. These fluctuations lead to a decrease in fringe visibility at the output, preventing one from operating in an error-free mode (see Ref. 21). Number eigenstates of course have zero-intensity fluctuations and thus enable error-free operation. However, in a real device some error in the output may be tolerable. A real problem might be phase noise added by the nonlinear medium itself. Such a noise source has been identified in certain nonlinear fibers.[17]

To model the effect of loss in the interferometer modes, we insert into each arm of the interferometer a beam splitter with transmittivity η, just before the output mirror. The required mode transformations now become

$$a_o = \hat{U}_L a_i - \hat{V}_L b_i + \hat{R}\,, \tag{14}$$

$$b_o = \hat{V}_L a_i - \hat{U}_L b_i + \hat{R}\,, \tag{15}$$

where

$$\hat{U}_L = \sqrt{\eta}\,\hat{U}\,, \tag{16}$$

$$\hat{V}_L = \sqrt{\eta}\,\hat{V}\,, \tag{17}$$

and $\hat{R}$ is a reservoir equation given by

$$\hat{R} = [(1-\eta)/2]^{1/2}(\hat{R}_1+\hat{R}_2)\,, \tag{18}$$

where $\hat{R}_1$ and $\hat{R}_2$ are independent reservoir operators describing fluctuations added by each beam splitter. These operators satisfy

$$[R_i, R_j^\dagger] = \delta_{ij}\,. \tag{19}$$

We further assume the reservoirs are at absolute zero (a reasonable assumption at optical frequencies) so that $\langle R_i^\dagger R_i\rangle=0$. When there is one photon in mode a_i, and all other modes are in the vacuum state, we find

$$\langle a_o^\dagger a_o\rangle = \tfrac{1}{2}\eta(1+\cos\theta)\,, \tag{20}$$

$$V(a_o^\dagger a_o) = \tfrac{1}{4}\eta^2\sin^2\theta + \tfrac{1}{2}\eta(1-\eta)(1+\cos\theta)\,. \tag{21}$$

When $\eta=1$, these equations reduce to Eqs. (6) and (8). When one photon is present in mode a_i and the control field,

$$\langle a_o^\dagger a_o\rangle = \tfrac{1}{2}\eta[1+\cos(\theta+\chi)]\,, \tag{22}$$

$$V(a_o^\dagger a_o) = \frac{\eta^2}{4}\sin^2(\theta+\chi) + \frac{\eta}{2}(1-\eta)[1+\cos(\theta+\chi)]\,. \tag{23}$$

At the ideal operating conditions ($\theta=0$, $\chi=\pi$) error-free operation is now no longer possible. Certainly in the case of Eqs. (22) and (23) the photon is never transmitted in mode a_o; however, it is not always transmitted in the case of Eqs. (20) and (21). Thus the absence of a photon at mode a_o does not necessarily imply a photon in the control beam.

The simple model discussed in this Letter shows that quantum uncertainties need place no limit on the accuracy of a Fredkin gate. However, even this nondissipative model can produce errors if not operated in the most advantageous way. If losses are included, the device must necessarily make mistakes from time to time; the greater the loss the more frequent the mistakes.

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