Quantum Disorder, Duality, and Fractional Statistics in 2+1 Dimensions

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We discover low-energy equivalence between two apparently unrelated Lagrangians with fractional statistics. Exploiting this equivalence, we are able to study the quantum disordered phase of the nonlinear σ model with Hopf term. We find that the quasiparticles in the disordered phase also have fractional statistics. There appears to be a dual relationship between the ordered and disordered phases.

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Recently there has been a great deal of interest in (2+1)-dimensional Lagrangians with Chern-Simons terms.[1–9] In particular, Dzyaloshinski, Polyakov, and Wiegmann[10] suggested that one such Lagrangian may be relevant to high-T_c superconductivity. In this paper, we show that two apparently unrelated Lagrangians with Chern-Simons terms are actually intimately connected with each other. Exploiting this equivalence to uncover the physics contained in these Lagrangians, we find that a duality-type relationship connects two different descriptions of the physics.

We begin by briefly reviewing what we need to define the Lagrangians in question. A few years ago, it was shown[2] that the soliton in the (2+1)-dimensional O(3) nonlinear σ model can be quantized to have fractional spin and statistics. This is accomplished by coupling a gauge potential A_μ to the topological current

$$J^\mu=(1/8\pi)\epsilon^{\mu\nu\lambda}\epsilon_{abc}n^a\partial_\nu n^b\partial_\lambda n^c \tag{1}$$

($a=1,2,3$ and $\mathbf{n}^2=1$), and by introducing a Chern-Simons term. Thus we write

$$\mathcal{L}_0=(2/g^2)(\partial\mathbf{n})^2+A_\mu J^\mu+\alpha\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}\,. \tag{2}$$

Integrating out A_μ in the Landau gauge $\partial_\mu A^\mu=0$, we find the action

$$S_0=\int d^3x(2/g^2)(\partial\mathbf{n})^2+\theta H\,, \tag{3}$$

where the so-called Hopf term

$$H=\frac{1}{4\pi}\int d^3x\,\epsilon^{\mu\nu\lambda}J_\mu\partial_\nu\frac{1}{\partial^2}J_\lambda \tag{4}$$

is nonlocal. We have defined

$$\theta=-1/8\alpha\,. \tag{5}$$

For a given spacetime configuration $\mathbf{n}(\mathbf{x},t)$ with the boundary condition $\mathbf{n}(\mathbf{x},t)\to$ some fixed $\mathbf{n}_0$ as $(\mathbf{x},t)\to\infty$, we have a map of S^3 into S^2. H is the normalized integral invariant associated with the homotopy $\pi_3(S^3)=Z$. Thus, H is an integer and we see clearly that classically the physics is independent of θ. Quantum mechanically, however, each spacetime history is associated in the path integral with a factor $e^{in\theta}$, where n labels the homotopy class of the history.[3,11] The parameter θ is clearly an angular variable and the physics described by (2) or (3) should be periodic in θ with period 2π.

The origin of fractional statistics[12] is quite simple. From (2) we have the equation of motion

$$2\alpha\epsilon^{\mu\nu\lambda}F_{\nu\lambda}=-J_\mu\,. \tag{6}$$

Consider a soliton sitting at rest and carrying q_0 units of the charge $\int d^2x J_0$. Far away from this soliton $F_{12}=0$ according to (6) and so the gauge potential A_μ is a pure gauge. However, it is topologically nontrivial since

$$\oint_C dx^iA_i=\int dS\,F_{12}=\frac{-1}{4\alpha}\int d^2x\,J_0=\frac{-q_0}{4\alpha}\,, \tag{7}$$

where C is a contour at infinity encircling the soliton. Taking another soliton slowly around this contour, we would thus induce in the wave function a phase proportional to $\oint dx^iA_i$. This phase can be interpreted as fractional statistics. It is important to realize that this argument does not depend on J^μ being the topological current in (1). Any conserved current would do in the construction. The localized charges associated with the current would then acquire fractional statistics. We note that the phase can also be computed from (4).

We may rewrite (2) and (3) in terms of a complex doublet

$$z=\begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

related to $\mathbf{n}$ by

$$n^a=z^\dagger\tau^a z\,, \tag{8}$$

where τ^a denote the Pauli matrices. The constraint $z^\dagger z=1$ implies that $\mathbf{n}^2=1$. (It is thus clear that z describes S^3 and $\mathbf{n}$ describes S^2. The relation between $\mathbf{n}$ and z defines the lowest nontrivial map of S^3 onto S^2.) Inserting into (2) we find

$$\mathcal{L}_1=(2/g^2)[\partial z^\dagger\partial z+(z^\dagger\partial z)^2]-(i/2\pi)A_\mu\epsilon^{\mu\nu\lambda}\partial_\nu z^\dagger\partial_\lambda z+\alpha\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}\,. \tag{9}$$

The calculation is most conveniently done by recognizing that J^μ is described by the two-form $\mathrm{tr}\,n(dn)^2$ with

$n \equiv n^a \tau^a = 2zz^\dagger - 1$. One finds easily that $\mathrm{tr} n(dn)^2 = 8 \times dz^\dagger dz$.

The use of z instead of $\mathbf{n}$ reveals that the topological current

$$J_\mu = \frac{-i}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu z^\dagger \partial_\lambda z = -\frac{i}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu (z^\dagger \partial_\lambda z)$$

is explicitly a curl. This fact is not at all clear when J_μ is written in terms of $\mathbf{n}$ as in (1). Referring back to (4), we see that the nonlocal Hopf term actually has three derivatives in the numerator. It is perhaps not surprising that two of these derivatives combine to cancel out the nonlocal operator $1/\partial^2$. The Hopf term when written in terms of z is actually local,

$$H = (i/4\pi^2) \int d^3x\, \epsilon_{\mu\nu\lambda} (z^\dagger \partial_\mu z) \partial_\nu (z^\dagger \partial_\lambda z) = (i/4\pi^2) \int (z^\dagger dz)(dz^\dagger dz) . \quad (10)$$

This result was derived in Ref. 3 using a slightly different formalism.

We see from (8) that the local transformation

$$z(x) \to e^{i\alpha(x)} z(x) \quad (11)$$

leaves $\mathbf{n}(x)$ invariant. This simply reflects the fact that we have used z with its 3 degrees of freedom (with $z^\dagger z = 1$) to describe $\mathbf{n}$ with its 2 degrees of freedom. The overall phase of z is in some sense a fictitious degree of freedom. We easily verify that $\mathcal{L}_1$ is indeed invariant under this local transformation as it must be since it is obtained from $\mathcal{L}_0$ by direct substitution. This local invariance can be made manifest [13-15] by introducing a gauge potential a_μ and by rewriting $\mathcal{L}_1$ as

$$\mathcal{L}_1' = (2/g^2) | (\partial_\mu - i a_\mu) z |^2 - (i/2\pi) A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu z^\dagger \partial_\lambda z + \alpha \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} . \quad (12)$$

As was emphasized earlier, the construction leading from (1) to (2) can be repeated for any (2+1)-dimensional theory with a conserved current J^μ. Thus, for instance, we can consider a charged scalar field ϕ and the Lagrangian[6,9]

$$\mathcal{L}_\phi = (2/g^2) | (\partial_\mu - i_\mu) \phi |^2 - V(\phi^\dagger \phi) + \beta \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} . \quad (13)$$

If we generalize this Lagrangian slightly by considering a doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

and if we suppose that the potential $V(\phi^\dagger\phi)$ forces $\phi^\dagger \phi = 1$, then we are led to consider a Lagrangian $\mathcal{L}_2$ somewhat reminiscent of $\mathcal{L}_1'$:

$$\mathcal{L}_2 = (2/g^2) | (\partial_\mu - i a_\mu) z |^2 + \beta \epsilon^{\mu\nu\lambda} a_\mu f_{\nu\lambda} . \quad (14)$$

Here we have purposely written a_μ in place of A_μ and defined $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. There has been a great deal of interest in this Lagrangian after Dzyaloshinski, Polyakov, and Wiegmann suggested that it may be relevant to high-T_c superconductivity.

One purpose of this Letter is to clarify the relation, if any, between $\mathcal{L}_1$ (or equivalently $\mathcal{L}_0$) and $\mathcal{L}_2$. At first sight, the two Lagrangians look totally different. In $\mathcal{L}_1$ the gauge potential is coupled to the topological current associated with how z or $\mathbf{n}$ is twisted, while in $\mathcal{L}_2$ the gauge potential is coupled to the current associated with the z quantum. In particular, the current in $\mathcal{L}_1$ involves two spacetime derivatives while the current in $\mathcal{L}_2$ involves only one derivative. This difference is accentuated when $\mathcal{L}_1$ is written as $\mathcal{L}_1'$, showing that there are two gauge potentials a_μ and A_μ which one could easily confuse.

Nevertheless, we will show that these two apparently distinct Lagrangians are closely related. Consider the Lagrangian

$$\mathcal{L}_{12} = \frac{2}{g^2} | (\partial_\mu - i a_\mu) z |^2 + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda + \alpha \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} . \quad (15)$$

The coefficient in front of the second term can be fixed by scaling A_μ. Integrating out A_μ we find [the manipulations[16] are the same as those leading to (10)]

$$\mathcal{L}_{12}' = \frac{2}{g^2} | (\partial_\mu - i a_\mu) z |^2 - \frac{1}{64\pi^2 \alpha} \epsilon^{\mu\nu\lambda} a_\mu f_{\nu\lambda} . \quad (16)$$

This is just $\mathcal{L}_2$ with

$$\beta = -1/64\pi^2 \alpha . \quad (17)$$

Associating angular parameters θ_α and θ_β with α and β as in (5), we find the inverse relation

$$\theta_\alpha / \pi = -\pi / \theta_\beta . \quad (18)$$

On the other hand, we can integrate out a_μ in $\mathcal{L}_{12}$. We find that

$$a_\mu = -[(1/16\pi) g^2 \epsilon_{\mu\nu\lambda} F_{\nu\lambda} + i z^\dagger \partial_\mu z] . \quad (19)$$

Substituting in $\mathcal{L}_{12}$, we find

$$\mathcal{L}_{12}'' = \frac{2}{g^2} [\partial_\mu z^\dagger \partial_\mu z + (z^\dagger \partial_\mu z)^2] - \frac{i}{2\pi} A^\mu \epsilon_{\mu\nu\lambda} \partial_\nu z^\dagger \partial_\lambda z + \alpha \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - \frac{g^2}{64\pi^2} F_{\mu\nu} F^{\mu\nu} . \quad (20)$$

Comparing with (9), we see that we can almost identify this Lagrangian with $\mathcal{L}_1$, except that it contains an extra Maxwell term F^2. However, the long-distance physics is unaffected by this Maxwell term since it has one more derivative than the Chern-Simons term. The long-distance behavior of the gauge potential around a soliton is the same regardless of whether we use (9) or (20). It is in this sense that $\mathcal{L}_1$ and $\mathcal{L}_2$ are "equivalent" at low energies.

We see from (18) that the "Chern-Simons coefficient" of the z quantum is the inverse of the Chern-Simons coefficient of the soliton made up of z quanta. This reciprocal relation is reminiscent of a relation in the hierarchical model.[17,18] Thus, if the soliton is a fermion ($\theta_a/\pi=1$), then so is the z quantum. In general, however, they would have different statistics.

It is intriguing to note that if we regard (18) as an iterative equation for the variable θ/π and if we restrict θ/π to lie in between -1 and $+1$, the fixed points are at $\theta/\pi=0$ and ±1. For instance, we have the sequence $\frac{5}{7}\to\frac{3}{5}\to\frac{1}{3}\to-1$. The physical significance of this purely mathematical result is not clear to us.

The physical properties of $\mathcal{L}_0$ and $\mathcal{L}_1$ have already been discussed in the literature.[2] We now study $\mathcal{L}_2$ directly to see how its properties are related to those of $\mathcal{L}_1$ (or $\mathcal{L}_0$).

Actually, the properties of $\mathcal{L}_2$ are closely related to those of $\mathcal{L}_\phi$, studied in Refs. 6, 8, and 9. The results obtained for $\mathcal{L}_\phi$ there also apply to $\mathcal{L}_2$. On long-distance scales, $\phi^\dagger\phi$ is effectively constrained to be 1. Classically, with $z^\dagger z=1$ the equation of motion for a_μ,

$$\tfrac{1}{4}\beta g^2\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda+a^\mu=-iz^\dagger\partial^\mu z\,, \tag{21}$$

indicates that a_μ has an effective mass given by $4/\beta g^2$ in the symmetry-broken vacuum $z=$const. Obviously this vacuum breaks the U(1) gauge symmetry associated with a_μ and the gauge potential becomes short ranged. As indicated in Ref. 6 the z particles behave as bosons in this vacuum state despite the Chern-Simons term.

The short range of a_μ presents an apparent paradox since we know from Ref. 2 that the soliton in $\mathcal{L}_0$ has fractional statistics. In $\mathcal{L}_2$ the soliton corresponds to the vortex given by

$$z(r,\varphi)=\begin{pmatrix}\cos f(r)e^{i\varphi}\\ \sin f(r)\end{pmatrix}, \tag{22}$$

where φ is the azimuthal angle and $f(r)$ satisfies $f(0)=\pi/2$, $f(\infty)=0$. One can easily check by calculating $\mathbf{n}=z^\dagger\tau z$ that the vortex in $\mathcal{L}_2$ is just the soliton in $\mathcal{L}_0$. One can also see this by noting that the toplogical charge of the solitons in $\mathcal{L}_1$ (or $\mathcal{L}_0$) is given by

$$\int d^2x\,\epsilon_{ij}\,\partial_i z^\dagger\partial_j z=\oint dl_i(\epsilon_{ij}z^\dagger\partial_j z)\,. \tag{23}$$

The right-hand side of (23) is just the winding number characterizing the vortex of $\mathcal{L}_2$. At infinity, even though $\mathbf{n}$ becomes some fixed number in the soliton solution, the z field has to twist around with the azimuthal angle φ. In (22), there is current flow at infinity given by $-iz^\dagger\partial_i z\to\partial_i\varphi$, thus generating a long-range pure-gauge tail for $a_i\to\partial_i\varphi$. This implies that the vortex has statistics determined by the phase of $e^{i8\pi^2\beta}=e^{-i/8\alpha}$, which is the same as that of the soliton in $\mathcal{L}_0$. The physics described by (6) and (21) are thus in some sense dual to each other: In one, flux is produced by a localized charge density while in the other, by current flow at infinity.

Since the cost in action of a fluctuation in the order parameter $\mathbf{n}$ or z is proportional to $1/g^2$ we expect a quantum disordered phase (i.e., a spin liquid phase) with the symmetry restored at large g^2. We can now exploit the equivalence of $\mathcal{L}_1$ and $\mathcal{L}_2$ to learn more about this phase. Actually, we are going to study a slightly generalized model, i.e., the CP^{N-1} model described by a Lagrangian $\mathcal{L}_1^N$ obtained from $\mathcal{L}_1$ by allowing z to be an N-component complex vector

$$z=\begin{pmatrix}z_1\\ \vdots\\ z_N\end{pmatrix}$$

satisfying $z^\dagger z=1$ and by replacing the 2 in the first term by N. The model discussed before corresponds to the case $N=2$ (CP^1 model). There are topological solitons in the CP^{N-1} model with z_1 and z_2 given by (22) and $z_i=0$ for $i>2$. The statistics of the soliton are again determined by the phase $e^{-i/8\alpha}=e^{i8\pi^2\beta}$.

To study the quantum disordered state we find the equivalent Lagrangian

$$\mathcal{L}_2^N=(N/g^2)\,|(\partial_\mu-ia_\mu)z|^2+\beta\epsilon^{\mu\nu\lambda}a_\mu f_{\nu\lambda}\,, \tag{24}$$

with $\beta=-1/64\pi^2\alpha$ more convenient. Applying the standard large-N expansion,[13-15] we find that when $g^2>g_c^2=\Lambda/4\pi$, the vacuum of the CP^{N-1} model is a quantum disordered state, i.e., $\langle z\rangle=0$, while for weak coupling g the vacuum breaks the SU(N) and the gauge symmetry ($\langle z\rangle=$const). Note that the nonlinear σ models in $1+2$ dimensions are not perturbatively renormalizable; they are well defined only with a finite ultraviolet cutoff. The critical coupling g_c is independent of β. The Chern-Simons term does not change the phase diagram, at least at this approximation. We may expect that even for small N the theory would still exhibit a symmetry-broken phase at small g^2 and a symmetry-restored phase at large g^2. However, these two phases may be separated by other phases whose range in g^2 vanish as $N=\infty$. (It is tempting to speculate that these phases may be dual to the small g^2 and large g^2 phases in the sense described below. The phase structure may be quite complicated, perhaps not dissimilar to the one discovered by Cardy and Rabinovici[19] in another context.)

The physical properties of the quantum disordered state are described by the following effective Lagrangian:[13-15]

$$\mathcal{L}_{\text{eff}}=(N/g^2)\,|(\partial_\mu-ia_\mu)z|^2-\lambda z^\dagger z-(N/4\gamma m)(f_{\mu\nu})^2+\beta\epsilon^{\mu\nu\lambda}a_\mu f_{\nu\lambda}\,, \tag{25}$$

where $m=\sqrt{\lambda}g^2/N$ is the mass of the z field, γ is a constant of order 1, and z is no longer subject to the constraint $z^\dagger z=1$. A Maxwell term f^2 is induced dynamically. If $\beta\neq0$ the gauge field is massive and does not confine. (For $\beta=0$ the Abelian gauge field $2+1$ dimensions is logarithmically confining.) Thus the z particles may appear in the physical spectrum and have fractional statistics given by the phase $e^{-i/8\beta}$ because in the disordered state the gauge symmetry is not broken. [The gauge-invariant Chern-Simons term produces the gauge boson's mass while the Maxwell term produces its kinetic energy. At long distance, the Maxwell term is unimportant compared to the Chern-Simons terms. In contrast, in (21) the Chern-Simons term produces the kinetic energy while gauge symmetry produces the mass.] Thus the phase transition between the symmetry-broken phase ($\langle z\rangle \neq 0$) and the quantum disordered phase ($\langle z\rangle = 0$) is a statistics-changing phase transition.[6]

We mention another intriguing but somewhat heuristic picture of the quantum disordered state. Let us start with the symmetry-broken vacuum $\langle z\rangle = \text{const}$ of the CP^{N-1} model and denote by φ the effective complex scalar field creating the soliton. (Strictly speaking, φ should carry an index labeling the different solitons.) At long distances, the soliton excitations in this vacuum are described by the following effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2} \,|\, (\partial_\mu + iA_\mu)\varphi \,|^2 + \alpha A_\mu F_{\nu\lambda} \epsilon^{\mu\nu\lambda} - M^2\varphi^2 - V(\varphi) \,, \tag{26}$$

where M is the soliton mass. The topological current in the CP^{N-1} model $J^\mu \sim \epsilon^{\mu\nu\lambda}\partial_\nu z^\dagger \partial_\lambda z$ corresponds to $\varphi^\dagger D^\mu \varphi$ in (26). φ has the same statistics as the soliton given by $e^{-i/8\alpha}$. When $g^2 > g_c^2$ presumably M^2 becomes negative and the symmetry-broken vacuum becomes unstable. The soliton field φ undergoes a condensation $\langle \varphi \rangle = \text{const}$ and the U(1) symmetry associated with the topological charge and A_μ is broken. The solitons disorder the orientation of z and the soliton-condensed state may correspond to the disordered state. The vortex in the soliton-condensed state has a statistic $e^{i8\pi^2\alpha} = e^{-i/8\beta}$ which corresponds to a z quantum in the disordered state.

We may thus have a nice dual description of the phases in $\mathcal{L}_1$ here. The symmetry-broken phase has a z condensation ($\langle z\rangle \neq 0$, $\langle\varphi\rangle = 0$). The soliton corresponds to a vortex in the z-condensed state. The quantum disordered phase has a soliton condensation ($\langle\varphi\rangle \neq 0$, $\langle z\rangle = 0$). The z quantum in the disordered phase corresponds to a vortex in the soliton-condensed state. This duality picture is reminiscent of the hierarchy scheme[17,18] in the fractional quantum Hall effect. Strictly speaking, (26) is not the dual theory of (24) because (26) only describes the low-energy properties of the topological current $-(i/2\pi)\epsilon^{\mu\nu\lambda}\partial_\mu z^\dagger \partial_\lambda z$, and does not include the SU(N) currents. It is only for $N=1$ that (26) is strictly a dual theory of (24).

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