Intermittency in Inverted-Pitchfork Bifurcations of Dissipative and Conservative Maps

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We consider intermittency in inverted-pitchfork bifurcation of a 1D dissipative map and of a 2D conservative map. Exact solutions to respective renormalization-group equations are constructed and scaling ratios obtained. The effect of noise is considered and scaling laws in the presence of noise are deduced. Results for the saddle-node bifurcation in 2D area-preserving maps are presented.

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Intermittency in dissipative dynamical systems is one of the more well-established routes to chaos. While intermittency of type 1 [1] has been widely studied for dissipative maps undergoing saddle-node bifurcation,[2-4] which is the generic codimension-one bifurcation with eigenvalue +1, no comparable investigation is to be found pertaining to the inverted-pitchfork bifurcation. The latter is the simplest nongeneric codimension-one bifurcation involving the eigenvalue +1 that displays intermittency.[5] This intermittency is expected to combine features of type 1 with those of type-2 and type-3 intermittencies,[1] because in this case an unstable branch of fixed points persists on the chaotic side of the bifurcation point.

Intermittency in conservative systems is another area worth investigating in view of its potential importance as an alternative route to chaos for Hamiltonian and other reversible systems. Intermittency in conservative maps has recently been shown to be relevant in describing inhomogeneous steady spatial structures in certain simple reaction-diffusion systems.[6]

In this context, we study in this paper, first, a 1D dissipative map and, next, a 2D area-preserving map, both exhibiting intermittency as a consequence of inverted-pitchfork bifurcation. We also include results on a 2D area-preserving map undergoing the saddle-node bifurcation because the latter is the generic codimension-one bifurcation of maps associated with eigenvalue +1.

The 1D dissipative map

$$x_{n+1}=(1+\epsilon)x_n+\beta x_n^3 \tag{1}$$

exhibits an inverted-pitchfork bifurcation at $\epsilon=0$, $x_n=0$. Converting the phase space to a circle by identifying the points $x=\pm S$, where S is a suitably chosen preassigned quantity, numerical iterations of the map have been seen to exhibit intermittency for small positive ϵ, while for ϵ negative, the fixed point $x=0$ attracts orbits initiated close to it. For sufficiently small $|\epsilon|$ and for regions of phase space close enough to the fixed point $x=0$, orbits may be computed from the differential equation approximation to (1), giving

$$1+\frac{\epsilon}{\beta x_n^2}=\left(1+\frac{\epsilon}{\beta x_{\text{in}}^2}\right)\exp(-2\epsilon n)\,, \tag{2a}$$

where $x_{\text{in}}=(x)_{n=0}$. Defining the laminar phase to correspond to $|x|\le x_0$ for some suitable x_0 ($<S$), defining $n(x_{\text{in}})$ as the value of n in Eq. (2a) for which $x_n^2=x_0^2$, and using a normalized white distribution for the initial points x_{in}, $P(x_{\text{in}})=1/2x_0$, the average time of passage in the laminar phase is given by $\langle l\rangle=\int_{-x_0}^{x_0}nP(x_{\text{in}})dx_{\text{in}}$; and on substituting from (2a) we obtain

$$\langle l\rangle=\frac{1}{x_0(\beta\epsilon)^{1/2}}\cot^{-1}\left[\frac{1}{x_0}\left(\frac{\epsilon}{\beta}\right)^{1/2}\right]. \tag{2b}$$

For $\epsilon/\beta\gg x_0^2$, Eq. (2b) gives

$$\langle l\rangle=1/\epsilon\,. \tag{2c}$$

Solving x_n from Eq. (2a) in terms of x_{in} and ϵ and defining the resulting function $f_\epsilon(x_{\text{in}};n)$ subject to the boundary condition $f'_{\epsilon=0}(0;n)=1$, we get

$$f_\epsilon(x;n)=\left(\frac{\epsilon}{\beta}\right)^{1/2}\left[\left(1+\frac{\epsilon}{\beta x^2}\right)\exp(-2\epsilon n)-1\right]^{-1/2}. \tag{3}$$

For each fixed n, this solves exactly the functional equation

$$f_\epsilon^2(x;n)=\alpha f_{\lambda\epsilon}(x/\alpha;n)\,. \tag{4}$$

[A class of exact continuous invertible solutions to Feigenbaum's functional equation for $\alpha\neq0$ has been constructed by McCarthy[7] furnishing simultaneously the stretching factor α ($=\sqrt{2}$) and the parameter scaling ratio λ ($=2$) (cf. Ref. 5).] In the following we shall encounter the interesting fact that the continuum approximation solves exactly the renormalization-group (RG) equation in 2D conservative maps as well. As a corollary to Eq. (4) we infer that the duration of the laminar phase of the intermittent orbits scales as $\epsilon^{-\nu}$ with $\nu(=\log2/\log\lambda)=1$. This $1/\epsilon$ scaling behavior is borne out by numerical computations (for not too small ϵ so that the inequality $\epsilon/\beta \gg x_0^2$ holds true) as shown in Fig. 1. The same value of ν might be obtained by other standard methods as well.[3,4] Apart from the time of duration of the laminar phase, we have numerically estimated the Lyapunov exponents[8] for orbits initiated close to the fixed point and have found these to be positive for small positive ϵ (thus signifying chaotic behavior) varying approximately as ϵ, as they should, in the limit $\epsilon \to 0$.

Since noise is ubiquitous in numerical experiments as well as in real systems, it is important to study the effect of noise on intermittency.[2-4,9] The differential equation approximation of (1) in the presence of noise reads (with n replaced by t),

$$dx/dt = \epsilon x + \beta x^3 + g\xi(t)\,, \tag{5}$$

where $\xi(t)$ is taken to be a Gaussian white noise and g measures the strength of the noise. The corresponding Kolmogoroff equation[2,10] describing the time evolution of the probability distribution $f(x,t)$ is

$$\partial f/\partial t = a\,\partial f/\partial x + \tfrac{1}{2} b\,\partial^2 f/\partial x^2\,, \tag{6}$$

where $a = \epsilon x + \beta x^3$, $b = g^2$. Assuming that the laminar phase corresponds to the region $-x_0 \le x \le x_0$ with x_0 sufficiently small, we can see from above that the mean time of passage $M(x_{\rm in})$ for a process starting at $x = x_{\rm in}$

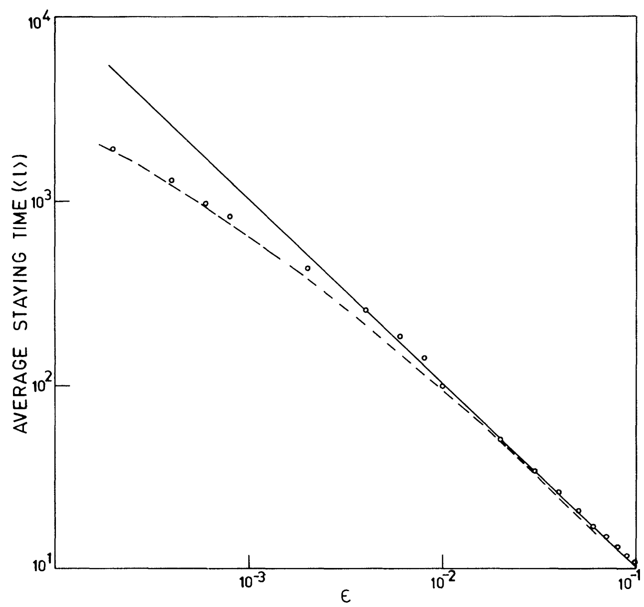

FIG. 1. Log-log plot of the average staying time $\langle l \rangle$ in the laminar phase vs ϵ. The mean has been taken over 2000 reentries in the laminar phase for every initial value and for 20 different initial values. The acceptance gate is $x_0 = \pm 0.01$, $S = 0.10$ (see text), and $\beta = 100.0$. The circles indicate the points obtained numerically. The dashed line refers to the theoretical curve obtained from the formula (2b) and the solid line to that from the asymptotic limit (2c).

at $t = 0$ satisfies

$$\tfrac{1}{2} g^2 d^2M/dx_{\rm in}^2 + (\epsilon x_{\rm in} + \beta x_{\rm in}^3)\,dM/dx_{\rm in} + 1 = 0\,. \tag{7}$$

Introducing the rescaling $y = x/\sqrt{\epsilon}$, $m = \epsilon M$, $\alpha = (g/\epsilon)^2$, we can rewrite (7) in the ϵ-free form

$$\tfrac{1}{2} \alpha^2 d^2m/dy_{\rm in}^2 + (y_{\rm in} + \beta y_{\rm in}^3)\,dm/dy_{\rm in} + 1 = 0\,, \tag{8}$$

from which the average length of the laminary phase

$$\langle l \rangle = \frac{1}{2x_0} \int_{-x_0}^{x_0} M(x_{\rm in})\,dx_{\rm in} \tag{9}$$

may be seen to be of the form

$$\langle l \rangle = (1/\epsilon) F(\alpha, \epsilon)\,, \tag{10}$$

where $F(\alpha,\epsilon) \to$ const for $\alpha \to 0$ and $F(\alpha,\epsilon) \sim \alpha^{-1/2}$ for $\alpha \to \infty$. An alternative approach involving a functional renormalization equation in the presence of noise[3,4,11,12] leads to the same conclusions.

The 2D area-preserving maps we shall consider will be of the form

$$x_{n+1} - 2x_n + x_{n-1} = f(x_n)\,, \tag{11a}$$

where $f(x_n)$ depends on some bifurcation parameter ϵ. Such maps can be cast into the de Vogelare form which has been fruitfully studied in the context of the period-doubling route to chaos in conservative maps.[13,14] We, however, rewrite (11a) as

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} x_n + f(x_n) + u_n \\ f(x_n) + u_n \end{pmatrix}, \tag{11b}$$

and shall assume the x-u phase space to be a 2D torus obtained by identifying opposite edges of a square of predetermined side S in $\mathbb{R}^2$ while choosing an appropriate metric which coincides locally with the Euclidean metric. Choosing

$$f(x) = \epsilon x + \beta x^3\,, \tag{11c}$$

we can easily verify that the map (11b) undergoes an inverted-pitchfork bifurcation at $\epsilon = 0$, $(x,u) = (0,0)$. For ϵ small and negative, the origin is an elliptic fixed point flanked by a pair of adjacent hyperbolic fixed points. It appears reasonable to surmise that the origin contains an invariant neighborhood comprising Kol'ogorov-Arnol'd-Moser curves.[15] For small positive ϵ, on the other hand, the origin is the only fixed point and is hyperbolic in nature and, as expected, numerical iterations of orbits initiated close enough to the origin show intermittency involving alternating laminar and chaotic phases. Since the map under consideration is two-dimensional, the functional renormalization-group equation corresponding to Eq. (4) will involve two universal functions $\phi_\epsilon(x,u)$ and $\psi_\epsilon(x,u)$ satisfying

$$\phi_\epsilon[\phi_\epsilon(x,u), \psi_\epsilon(x,u)] = (1/a)\phi_{\lambda\epsilon}(ax,bu)\,, \tag{12a}$$

$$\psi_\epsilon[\phi_\epsilon(x,u), \psi_\epsilon(x,u)] = (1/b)\psi_{\lambda\epsilon}(ax,bu)\,, \tag{12b}$$

where a and b are universal coordinate stretching ratios and λ is the parameter stretching ratio characterizing the scaling. Relying on our earlier procedure involving the role of the differential equation approximation in leading to the exact solution to the RG equation characterizing the transition to chaos, we consider

$$d^2x/dt^2=\epsilon x+\beta x^3 . \tag{13}$$

Referring to the first and the second integrals of this equation, we consider the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} \to P_\epsilon \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} \phi_\epsilon(x,u) \\ \psi_\epsilon(x,u) \end{pmatrix}, \tag{14a}$$

where ϕ_ϵ and ψ_ϵ are defined through the equations

$$\int_x^{\phi_\epsilon} \frac{dq}{[\epsilon q^2+\frac{1}{2}\beta q^4+(u^2-\epsilon x^2-\frac{1}{2}\beta x^4)]^{1/2}}=t \tag{14b}$$

$$\psi_\epsilon^2-\epsilon\phi_\epsilon^2-\tfrac{1}{2}\beta\phi_\epsilon^4=u^2-\epsilon x^2-\tfrac{1}{2}\beta x^4 . \tag{14c}$$

For given ϵ, Eqs. (14a)-(14c) actually define a one-parameter family of maps characterized by t. For each fixed t, we observe that ϕ_ϵ and ψ_ϵ defined by (14b) and (14c) do indeed satisfy the RG equations (12a) and (12b) for a particular set of values of a, b, and λ. To see this we note that (12a) and (12b) may be written from (14a) as

$$P_\epsilon\left[P_\epsilon\begin{pmatrix} x \\ u \end{pmatrix}\right]=\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} P_{\lambda\epsilon}\begin{pmatrix} ax \\ bu \end{pmatrix} . \tag{15}$$

Denoting the left-hand side of this equation by $\binom{\xi}{\eta}$, we obtain from (14b) and (14c)

$$\eta^2-\epsilon^2\xi^2-\tfrac{1}{2}\beta\xi^4=\psi_\epsilon^2-\epsilon\phi_\epsilon^2-\tfrac{1}{2}\beta\phi_\epsilon^4 , \tag{16a}$$

$$\int_{\phi_\epsilon}^{\xi}\frac{dq}{[\epsilon q^2+\frac{1}{2}\beta q^4+(\psi_\epsilon^2-\epsilon\phi_\epsilon^2-\frac{1}{2}\beta\phi_\epsilon^4)]^{1/2}}=t . \tag{16b}$$

Equation (14c) shows that the integrands in Eqs. (14b) and (16b) are identical, and hence,

$$t=\frac{1}{2}\int_x^{\xi}\frac{dq}{[\epsilon q^2+\frac{1}{2}\beta q^4+(u^2-\epsilon x^2-\frac{1}{2}\beta x^4)]^{1/2}} . \tag{17}$$

On the other hand, if Eq. (15) is to hold we must have

$$b^2\eta^2-\lambda\epsilon a^2\xi^2-\tfrac{1}{2}\beta a^4\xi^4 = b^2u^2-\lambda\epsilon a^2x^2-\tfrac{1}{2}\beta a^4x^4 \tag{18a}$$

$$t=\int_{ax}^{a\xi}\frac{dq}{[\lambda\epsilon q^2+\frac{1}{2}\beta q^4+(b^2u^2-\lambda\epsilon a^2x^2-\frac{1}{2}\beta a^4x^4)]^{1/2}} . \tag{18b}$$

Equations (14c), (16a), and (17) show that (18a) and (18b) are indeed satisfied with

$$a=2 , \tag{19a}$$

$$b=4 , \tag{19b}$$

$$\lambda=4 . \tag{19c}$$

Thus, here too we arrive at an exact solution of the RG equation and obtain the scaling ratios by referring to the solution of the differential equation approximation (13) to the original map (11b) and (11c). Equation (19c) implies that the average duration of the laminar phase should scale as

$$\langle l\rangle \sim \epsilon^{-1/2} . \tag{20}$$

The effect of noise can be taken care of in this case by including a white-noise source in Eq. (13), leading finally to the Kolmogoroff equation

$$\frac{\partial f}{\partial t}=a_1\frac{\partial f}{\partial x}+a_2\frac{\partial f}{\partial u}+\frac{1}{2}\sum_{i,k=1}^{2} b_{ik}\frac{\partial^2 f}{\partial r_i\,\partial r_k} , \tag{21}$$

where $f=f(x,u,t)$, $r_1=x$, $r_2=u$, and the coefficients are $a_1=u$, $a_2=\epsilon x+\beta x^3$, $b_{11}=b_{12}=0$, $b_{22}=g^2$ (g being the strength of the noise as before). The mean time of passage $M(x_{\rm in},u_{\rm in})$ of the process initiated at $(x_{\rm in},u_{\rm in})$ satisfies

$$a_1\frac{\partial M}{\partial x_{\rm in}}+a_2\frac{\partial M}{\partial u_{\rm in}}+\frac{1}{2}b_{22}\frac{\partial^2 M}{\partial u_{\rm in}^2}+1=0 . \tag{22}$$

Rescaling according to

$$x=\sqrt{\epsilon}X,\ \ u=\epsilon U,\ \ M=m/\sqrt{\epsilon},\ \ \alpha=g^2\epsilon^{-5/2} , \tag{23}$$

we get the ϵ-free equation

$$U\frac{\partial m}{\partial x}+(X+\beta X^3)\frac{\partial m}{\partial U}+\frac{1}{2}\alpha\frac{\partial^2 m}{\partial U^2}+1=0 . \tag{24}$$

Assuming the laminar phase to correspond to points within the region $-x_0\le x_{\rm in}\le x_0$, $-u_0\le u_{\rm in}\le u_0$ for some suitably defined small x_0,u_0, the average (over all initial values) time of duration of the laminar phase

$$\langle l\rangle=\frac{1}{4x_0u_0}\int_{-x_0}^{x_0}\int_{-u_0}^{u_0} M\,dx_{\rm in}\,du_{\rm in} \tag{25a}$$

may be seen from above to be of the form

$$\langle l\rangle=(1/\sqrt{\epsilon})F(\alpha,\epsilon) , \tag{25b}$$

i.e., $\sqrt{\epsilon}\langle l\rangle$ is, approximately, some universal function of $g^2/\epsilon^{5/2}$. We can reasonably surmise the asymptotic form $F\to$ const for $\alpha\to 0$, corresponding to the scaling law $\langle l\rangle\sim\epsilon^{-1/2}$ in the absence of noise.

We conclude the discussion on 2D area-preserving maps by quoting the corresponding results for the saddle-node bifurcation. For this we consider the map (11b) with

$$f(x)=\epsilon+\beta x^2\ \ (\beta>0) . \tag{26}$$

Here also the solution to the differential equation approximation to the map furnishes an exact one-parameter family of solutions to the RG equations (12a) and (12b), namely,

$$\tfrac{1}{2}\psi_\epsilon^2-\epsilon\phi_\epsilon-\tfrac{1}{3}\beta\phi_\epsilon^3=\tfrac{1}{2}u^2-\epsilon x-\tfrac{1}{3}\beta x^3\,, \quad (27a)$$

$$t=\int_x^{\phi_\epsilon}\frac{dq}{[\epsilon q+\frac{1}{3}\beta q^3+(\frac{1}{2}u^2-\epsilon x-\frac{1}{3}\beta x^3)]^{1/2}}\,, \quad (27b)$$

providing us simultaneously with the scaling ratios

$$a=4,\ b=8,\ \lambda=16\,. \quad (28)$$

The scaling equation for $\langle l\rangle$ comes out in this case to be

$$\langle l\rangle=\epsilon^{-1/4}F(g^2\epsilon^{-7/4},\epsilon)\,. \quad (29)$$

Intermittency in saddle-node bifurcations of dissipative maps has been identified in diverse physical systems.[16,17] It may be interesting to look for intermittency in inverted-pitchfork bifurcation which requires the presence of special symmetry in the systems under consideration. (For intermittency in several other contexts see, e.g., Refs. 18–21). Numerical aspects of intermittency in two-dimensional conservative maps discussed here, namely, staying times in the laminar phase, Lyapunov exponents, and power spectra are being investigated.

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