Stability of a Moving Hypercooled First-Order Phase Front: Application to the Superfluid ^{3}He A-B Interface

John Palmeri
Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801
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The linear stability of a moving first-order phase boundary in the presence of *hypercooling* and *anisotropy* is investigated. Applying the results to the A-B transition leads us to conclude that in the temperature regime already explored experimentally the moving planar interface should be linearly stable; we also find that a new theoretical prediction for the terminal velocity of the planar phase boundary is in good quantitative agreement with experiment.

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The dynamics and instabilities of first-order phase fronts are of interest in a variety of settings in many areas of physics. There is a special class of first-order phase transitions known as *hypercooled,* where the latent heat released during the transition is no longer able to reheat the boundary region back above the thermodynamic transition temperature. Such is the case, for example, for the martensitic transition in some solids, the solid-superfluid transition in ^{4}He, possibly for some liquid-crystal phase transitions, and—probably most spectacularly—for the A-B transition in superfluid ^{3}He. In these cases the flow of latent heat no longer plays a limiting role in determining the velocity of the transition, and the well-known Mullins-Sekerka instability of a planar interface[1] becomes inoperative. Consequently, some microscopic mechanisms must be sought to explain the speed of hypercooled transitions and the stability question must be reexamined.

In this Letter we show that a planar, hypercooled first-order phase front propagating in the presence of anisotropy is susceptible to a new type of instability that depends entirely on the existence and strength of the anisotropy. These results are then used to investigate the stability of a planar ^{3}He A-B interface moving at its terminal velocity (with the liquid-crystal-like orbital $\hat{\boldsymbol{l}}$ vector in the A phase playing the role of an anisotropy axis).

The A and B phases are symmetry unrelated, degenerate—or nearly degenerate—bulk states of liquid ^{3}He at mK temperatures,[2] and the dynamics of the first-order phase transition between these two distinct Fermi superfluids provides a novel and experimentally accessible forum for studying nonequilibrium interface and nucleation phenomena.[3] Since the $A \rightarrow B$ transition can be substantially hypercooled,[2,4] the dynamics of the topologically stable phase boundary can be studied experimentally over a large region of undercooling.[3] Leggett and Yip (LY),[5,6] Markelov,[7] and Kopnin[8] have initiated the theoretical study of the dynamics of the A-B interface. LY treat the A-B phase boundary simply as a spatial modulation of the Cooper pair wave function and find [for T not too close to $T_{AB}(P)$, the coexistence temperature] that the important friction mechanism governing the dynamics of the A-B interface at not too low temperatures is the Andreev scattering of the normal excitations by the moving gap distortion, while at lower temperatures (roughly $T/T_c < 0.5$, with T_c the normal-to-superfluid second-order transition temperature) the friction mechanism is dominated by the Cooper pair breaking induced by the distorted, time-varying self-consistent pair potential. The thermodynamic driving force on the moving interface is taken to be $\Delta G_{AB}(P,T,H)$, the difference in Gibbs free energy density between the two bulk phases evaluated at the pressure P and the temperature $T < T_{AB}$ of the hypercooled A phase.[5,6] Furthermore, we assume that all hydrodynamic fluxes in the bulk phases vanish[6] (cf. Ref. 7).

LY have studied the mobility of the uniformly moving planar interface and their calculation of the friction coefficient due to Andreev scattering, Γ_A, leads to a prediction for the terminal velocity, $v_{AB} = \Delta G_{AB}/\Gamma_A$, of the moving interface that is in reasonable qualitative agreement with the experimental results. The theory, however, underestimates v_{AB} by roughly a factor of 2, and this discrepancy has led to the suggestion[5] that due to the sensitive dependence of Γ_A on $\mu(\mathbf{r}) \equiv \hat{\boldsymbol{l}}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r})$—where $\hat{\mathbf{n}}(\mathbf{r})$ is the local normal to the interface directed towards the B phase, and $\mathbf{r} \in$ interface—the moving planar interface may be dynamically unstable and perhaps spontaneously *roughen;* the roughened interface might then propagate with an effective velocity greater than the one predicted for the planar interface. Another possible reason for this discrepancy is that a wrong value for the cutoff in a logarithmically divergent integral over quasiparticle (QP) momenta was used in the calculation of Γ_A.[5,6]

Since we argue below that in the currently accessible experimental regime the moving planar interface is linearly stable, it is unlikely that the discrepancy between theory and experiment is due to the presence of a *roughened* interface. The source of the discrepancy most likely lies in the wrong choice for the QP momentum cutoff,[6] and we demonstrate below that for the textural

boundary conditions between the A and B phases expected for the Los Alamos (LANL) experiment[3] a calculation of Γ_A with the momentum cutoff properly taken into account gives good quantitative agreement with the data.

A thermal fluctuation can give rise to a bulge in the moving planar interface, which in the presence of anisotropy may in turn lead to a friction coefficient and surface energy that depend on position through $\mu(\mathbf{r})=\hat{\boldsymbol{l}}\cdot\hat{\mathbf{n}}(\mathbf{r})$, leading to the possibility that the bulge gets distorted and continues to grow. [We assume that the anisotropy $\hat{\boldsymbol{l}}$ field is not only stationary, but also uniform in space (see below).] This type of growth is opposed by the action of an *effective* surface tension, which (usually) tends to corral any bulges in the interface. The stability of the planar interface will clearly depend in some way on a competition between the magnitude of this effective surface tension and how rapidly Γ varies with μ.

We treat the interface in the thin-wall approximation,[5,6] which means that the fluctuations about the planar interface are assumed to have a characteristic wavelength $\lambda \gg d(T)\equiv c\xi(T)$, where d is the interface width, $\xi(T)$ is the temperature-dependent coherence length, and c is a constant ~ 5–10.[9,10] In this approximation, the interface can be described by a surface $z=q(x,y;t)$ (collective coordinate description), with the z axis taken along the tube and z measured from some reference plane fixed to the cell walls to some reference point on the interface.

Alternatively, the motion of the interface can be described by a differential displacement vector field $\mathbf{u}(\mathbf{r},t)$. At time t the interface is taken to consist of the locus of points $S(t)=\{\mathbf{r}:\ \mathbf{r}(x,y;t)=(x,y,q(x,y;t))\}$. After an infinitesimal time interval dt a point $\mathbf{r}$ on the interface will undergo a differential displacement $\mathbf{u}(\mathbf{r})=u\hat{\mathbf{n}}(\mathbf{r})$ normal to the interface at $\mathbf{r}$ to a new point $\mathbf{r}'$ with $\mathbf{u}(\mathbf{r},t)\equiv\mathbf{r}'-\mathbf{r}$ and $\mathbf{r}'\in S(t+dt)$; only perpendicular displacements are physical, since tangential motions simply give rise to a physically unimportant reparametrization of the interface (cf. Ref. 11).

To consider perturbations about a reference planar interface moving at its terminal velocity v_{AB}, we assume that at time $t=0$ the interface differs only slightly from a steadily moving plane with normal $\hat{\mathbf{n}}=-\hat{\mathbf{z}}$. The initial interface surface is described by $z=w_0(x,y)\equiv q(x,y;t=0)$ [with $|\nabla_{(2)}w_0(x,y)|\ll 1$ where $\nabla_{(2)}\equiv(\partial_x,\partial_y)$]. For $t>0$ the position of the interface is given by $z=q(q,y;t)=v_{AB}t+w(x,y;t)$ with w considered to be a small perturbation.

An *effective* equation of motion for $q(x,y;t)$ can be obtained using the Lagrangian theory of two-dimensional membranes.[12] The surface $z=q(x,y;t)$ is defined for $(x,y)\in\Sigma$, the base of the tube, and the B phase occupies the region with $z<q(x,y;t)$. In the following analysis edge effects due to the walls of the tube will be entirely neglected[13]: Σ is taken to be infinite, and the solutions to the equation of motion need only remain bounded at infinity. The Lagrangian density $\mathcal{L}=\mathcal{L}(x,y;q,\partial_t q,\partial_\alpha q,\partial_\alpha^2 q)$ (with $\alpha=x,y$) takes the form[13]

$$\mathcal{L}=\tfrac{1}{2}M^*(dS/dA)(\partial_t\mathbf{u}[q])^2 - \sigma_{AB}[q](dS/dA)+q\,\Delta G_{AB}\,,$$

where $dS=[1+(\nabla_{(2)}q)^2]^{1/2}dA$ (with $dA=dx\,dy$) is the actual area element at the point $\mathbf{r}$ on the surface, $M^*[q]$ is the effective areal mass density for the interface, $\mathbf{u}[q]$ is taken as a functional of q, and $\sigma_{AB}[q]$ is the surface energy density. To account for energy dissipation, we introduce a Rayleigh dissipation function

$$\mathcal{F}[q]=\tfrac{1}{2}[1+(\nabla_{(2)}q)^2]^{1/2}\Gamma[q](\partial_t\mathbf{u}[q])^2\,,$$

which in general is a functional of q and is defined so that the total rate of energy dissipation is $dE/dt=-2\int\int_\Sigma dA\,\mathcal{F}[q]$.

The requirement that the variation of the action about the actual motion vanish implies that $q(x,y;t)$ must satisfy the following generalized Euler-Lagrange equation:

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t q)}\right)-\partial_\alpha^2\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha^2 q)}\right)+\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha q)}\right)-\frac{\partial\mathcal{L}}{\partial q}+\frac{\partial\mathcal{F}}{\partial(\partial_t q)}=0 \qquad (1)$$

(summation convention with $\alpha=x,y$).

For a static planar interface the value of the surface energy density σ_{AB} depends in a complicated way on the textural boundary conditions between the A and B phases.[10,14,15] When the interface is moving it is a good approximation for all but the slowest speeds to completely neglect the dynamics of the $\hat{\boldsymbol{l}}$ vector (due to the orbital viscosity) and assume that the A-phase $\hat{\boldsymbol{l}}$ texture remains frozen in its initial (and not necessarily minimum energy) configuration.[5,6] The response of the other interfacial textural degrees of freedom—connected with the other symmetry-breaking variables in the bulk phases—to the passage of the interface is an involved problem, and it is not possible to treat this topic in depth here[13]; instead, we make the plausible simplifying assumption that at each point $\mathbf{r}$ on the moving interface the other interfacial degrees of freedom adjust themselves adiabatically to minimize the surface energy density for the local value of $\mu(\mathbf{r})=\hat{\mathbf{n}}(\mathbf{r})\cdot\hat{\boldsymbol{l}}$ (*adiabaticity assumption*). It is then plausible (cf. Ref. 14) that σ_{AB} should depend on q only through $\mu(\mathbf{r})$ [via $\hat{\mathbf{n}}(\mathbf{r})$, see below] and the extrinsic curvature $\kappa(\mathbf{r})$, so that for a slightly curved interface[13]

$$\sigma_{AB}[\mu(\mathbf{r}),\kappa(\mathbf{r})]\approx\sigma_0[1+a\delta\mu+\tfrac{1}{2}b(\delta\mu)^2+\tfrac{1}{2}c(\kappa d)^2]\,,$$

where σ_0 is the surface energy density of the moving reference planar interface (for which $\mu=\mu_0\equiv-\hat{\boldsymbol{l}}\cdot\hat{\mathbf{z}}$),

$\delta\mu(\mathbf{r}) = \mu(\mathbf{r}) - \mu_0$, $a \equiv (1/\sigma_0)(\partial\sigma_{AB}/\partial_\mu)_0$, $b \equiv (1/\sigma_{j0}) \times (\partial^2\sigma_{AB}/\partial\mu^2)_0$, and c are constants presumably of order unity, and κ is given by

$$\kappa(\mathbf{r}) = \kappa(x,y) = \nabla_{(2)} \cdot [(1 + |\nabla_{(2)} q|^2)^{-1/2} \nabla_{(2)} q(x,y)] .$$

Using

$$\hat{\mathbf{n}}(\mathbf{r}) = \hat{\mathbf{n}}(x,y) = (\partial_x q, \partial_y q, -1)(1 + |\nabla_{(2)} q|^2)^{-1/2}$$

for the normal, we find

$$\delta\mu(x,y) \approx \hat{l} \cdot \nabla_{(2)} q(x,y;t) + \tfrac{1}{2} \hat{l}_z (\nabla_{(2)} q)^2 .$$

We assume that for an interface only weakly distorted from a plane the friction coefficient $\Gamma(\mu)$ can be expanded in powers of $\delta\mu(\mathbf{r})$; keeping terms up to first order yields $\Gamma(\mu) \approx \Gamma_0 + \Gamma_0' \delta\mu(\mathbf{r})$, where $\Gamma_0 \equiv \Gamma(\mu_0)$ and $\Gamma_0' \equiv (\partial\Gamma/\partial\mu)_{\mu=\mu_0}$ (thus $v_{AB} = \Delta G_{AB}/\Gamma_0$).

A linear equation of motion for small perturbations $w(x,y;t)$ about a planar interface moving steadily at its terminal velocity v_{AB} can be derived using a quadratic approximation to $\mathcal{L}$ and approximating the Rayleigh dissipation function by $\mathcal{F} \approx \frac{1}{2}\Gamma(\mu)[\partial_t q(x,y;t)]^2$ with $\Gamma(\mu(\mathbf{r}))$ expanded to first order in $\delta\mu(\mathbf{r})$; using Eq. (1) we find

$$M_0^* \partial_t^2 w + \Gamma_0 \partial_t w - \sigma_0 b (\hat{l} \cdot \nabla_{(2)})^2 w - \sigma_0 [1 + a\hat{l}_z] \nabla_{(2)}^2 w + v_{AB} \Gamma_0' (\hat{l} \cdot \nabla_{(2)}) w + c d^2 \sigma_0 \nabla_{(2)}^4 w = 0 . \quad (2)$$

This equation should be valid for perturbations with wave vectors $k = |\mathbf{k}|$ in the range $k_{\min} \le k \le k_{\max} \ll d^{-1}$, where $k_{\min}^{-1} \sim$diameter of the experimental tube.

For a perturbation with wave vector $\mathbf{k} = k\hat{\mathbf{k}}$, we first define two crucial quantities, $f(\hat{\mathbf{k}}) \equiv 1 + b(\hat{l} \cdot \hat{\mathbf{k}})^2 + a\hat{l}_z$ and

$$\Lambda(\hat{\mathbf{k}}) \equiv (M_0^* v_{AB}^2/\sigma_0)(\Gamma_0'/\Gamma_0)^2 (\hat{l} \cdot \hat{\mathbf{k}})^2 ,$$

which play the roles of dimensionless restoring (due to surface tension) and destabilizing (due to anisotropy) forces (normalized by $k^2\sigma_0$) on the perturbation and therefore determine the stability properties of the moving planar interface.

Then by Fourier analyzing Eq. (2) we arrive, after some work, at our principal (and surprisingly simple) conclusion that *the steadily moving planar interface is linearly stable* if and only if $\Lambda(\hat{\mathbf{k}}) \le f(\hat{\mathbf{k}})\ \forall\ \hat{\mathbf{k}}$.[13]

To estimate the value of Λ for the A-B phase front in the currently accessible region of the (P,T) phase diagram, we need an expression for the friction coefficient Γ_A in the temperature range where the Andreev friction mechanism dominates. At low temperatures only quasiparticles in the A phase with momenta in small cones around $\pm\hat{l}$ in QP momentum space have low enough energy to be thermally excited, and a simple low-temperature form for the Andreev friction coefficient, valid when $T \ll \Delta_B(T)$ (the B-phase energy gap), can be found from the general expression in Ref. 5 (see Ref. 13 for details). Here we use the simple low-T forms for Γ_A, which turn out to be reasonably good approximations for all $T < T_{AB}$ (at least near melting pressure), although they slightly overestimate Γ_A (and therefore lead to an underestimate for v_{AB}, see Fig. 1) at higher T.

Using these "low-T" forms for Γ_A, together with both the experimentally measured and the theoretically estimated values for v_{AB}, and the rough estimate $M_0^* \sim \sigma_0/v_F^2$ for the inertial mass (which is due to the intrinsic inertia associated with the time-dependent deformation of the ^{3}He order parameter), we can estimate Λ (where v_F is the Fermi velocity). Furthermore, under the adiabaticity assumption, it can be shown that $f \sim 1$ (i.e., $f \ll 1$) $\forall\ \hat{\mathbf{k}}$.[13] In the evaluation of Γ_A (and therefore Λ) as a function of μ it is necessary to consider two regions separated by a small cutoff μ_c: (1) for $\mu \gg \mu_c \sim (\Delta_B/\epsilon_F)^{1/2} \sim 0.05$, (where $\epsilon_F \sim$ Fermi energy ~ 1 K) and low T, Γ_A is independent of the cutoff, and we find $\Lambda \ll 1$, implying that the A-B interface is always stable. This conclusion becomes increasingly marginal at low T ($T/T_c < 0.6$) for $\mu \to \mu_c$ as v_{AB}/v_F increases rapidly in the Andreev regime (before saturating in the pair-breaking regime). The above cutoff, which should be valid for roughly $T/T_{AB} > 0.6$ at melting pressure,[13] is intrinsic to the Andreev friction mechanism and can be determined by inspecting the scattering solutions to the Bogoliubov–de Gennes equations in the presence of a static A-B boundary. (The cutoff arises because the condition for Andreev reflection cannot be satisfied for

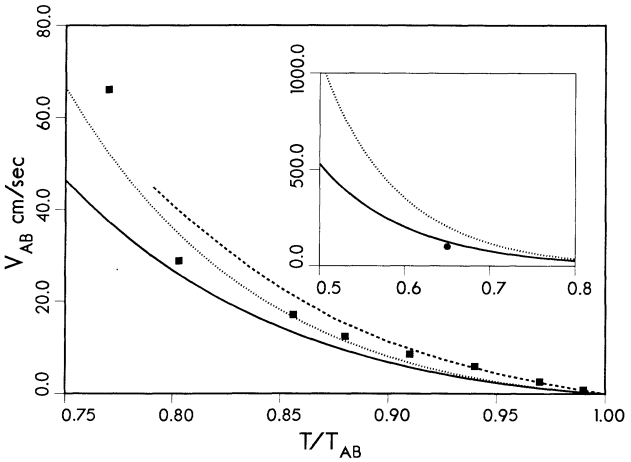

FIG. 1. Terminal velocity v_{AB} of the planar phase boundary as a function of $T/T_{AB}(P)$. Square data points (from Ref. 2) are at pressure $P \approx 33.6$ bars and field $H \approx 100$ G. Solid and dotted curves are the "low-T" results for $\mu = 0$ and 1 at $P \approx 34$ bars and $H = 0$. Dashed curve is the full LY result (Ref. 5) for $\mu = 1$ at melting pressure and $H = 0$. Inset: Circular data point (from Ref. 18) is for $P \approx 30$ bars. [For the theoretical curves ΔG_{AB} is taken from experiment (cf. Refs. 5 and 6).]

QP momenta too close to the plane of the interface.[8,13]) (2) For $\mu=0$ (which, due to the influence of the cell walls and the applied magnetic field on the $\hat{l}$ vector, should hold in the LANL experiments[3]) and low T, $\Gamma_A(\mu=0)$ depends logarithmically on the inverse of the cutoff μ_c, and we find $\Gamma_A'(\mu=0)=0$; since Λ is then zero the interface is again linearly stable. Interestingly enough, there are already experimental indications that something bizarre is going on at very low T (near the crossover to the pair-breaking regime, which is outside the scope of our analysis),[16] and one possible explanation is that the interface has roughened.

We can see (Fig. 1) that the low-T forms for Γ_A are really quite good by comparing our low-T prediction for the terminal velocity v_{AB} of the planar interface at $\mu=1$ with the results obtained by LY[5] using their full expression for Γ_A (which for $\mu=1$ should be insensitive to the incorrect cutoff in QP momentum space used in Ref. 5 since the important QP states are clustered far from the cutoff region). If we assume that the low-T result for v_{AB} evaluated at $\mu=0$ slightly underestimates v_{AB} in the same way it does at $\mu=1$ then—except for the data point at $T/T_{AB}\approx 0.77$, where v_{AB} seems anomalously large—our *adjusted* theoretical prediction for v_{AB} at $\mu=0$ is in good quantitative agreement with experiment (with essentially no adjustable parameters); lower-T data are desirable to check this agreement more carefully.

In conclusion, we have found that due to the presence of anisotropy the planar A-B interface is susceptible to a linear instability, but for the range of relevant parameters explored so far[3] the interface is within the instability threshold. In addition, the good agreement between our new theoretical prediction for v_{AB} and experiment provides convincing (albeit indirect) quantitative evidence for the existence of the subtle Andreev scattering processes in ^{3}He. Finally, perhaps the new dynamical instability discussed above can be observed in other systems (e.g., the nematic-isotropic interface of a liquid crystal, see Ref. 17).

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