PHYSICAL REVIEW LETTERS

VOLUME 62 | 17 APRIL 1989 | NUMBER 16

Analytic Evaluation of the Multifractal Properties of a Newtonian Julia Set

M. Nauenberg and H. J. Schellnhuber[a]

Institute of Nonlinear Science, University of California at Santa Cruz, Santa Cruz, California 95064

(Received 7 July 1988)

The Julia set associated with the Newtonian map for the solution of $z^3-1=0$ in the complex plane is investigated by purely analytic means. The hierarchical organization of this strange repeller is explained from basic principles. The family of generalized fractal dimensions and the equivalent spectrum of scaling indices is evaluated using an extension of the Hentschel-Procaccia renormalization scheme. Our quantitative results are in excellent agreement with direct numerical computations.

PACS numbers: 05.45.+b, 02.30.+g, 47.20.Tg

During the past few years, a general formalism has been developed to quantify the multifractal properties of complex sets which occur in nonlinear physics.[1,2] Familiar examples are strange attractors associated with chaotic dynamical systems, and Julia sets which are the boundaries of basins of attractors of iterated maps in the complex plane C.[3] While considerable progress has been made in calculating the generalized fractal dimensions of 1D strange attractors, very few analytic results exist for Julia sets[4] and numerical investigations tend to be extremely tedious for these intricate objects.[5]

This is regretable because iterated complex maps and the Julia sets organizing their dynamics are not only fascinating mathematical systems, but relevant to physics in various ways: Applying the powerful tools of analytic calculus to nonlinear complex maps one can gain, for example, a basic understanding of the notorious small-denominator problem plaguing Hamiltonian mechanics,[6] or of the scaling behavior of circle maps[7] modeling successfully the onset of chaos in many real physical situations.

Julia sets, in particular, are employed by Ruelle as paradigms for "mixing repellers" (see Ref. 4), while Kadanoff considers them as models for "strange repellers of dynamical systems"[8] arising, e.g., in the context of electron motion in quasiperiodic potentials.[9] Direct use of Julia-set theory is made in the analysis of the spectral and localization properties of quantum states in hierarchical tight-binding models.[10] Other applications, for example, to the Yang-Lee theory of phase transitions, to magnetic spin models, and to diffusion-limited aggregation (see below) have also been discussed recently.

In this paper we evaluate analytically all the dimensions of the famous Julia set which arises by iterating in C Newton's map for the solution of the equation $z^3-1=0$. The problem of determining this Julia set, which constitutes the boundary of the basins of attraction of the cubic roots of unity, was first formulated by Cayley in 1879.[11] Remarkably, it remained unsolved until 1977 with the application of computer graphics to this problem first by Hubbard,[12] and later on by many others.[3,13-15]

Nevertheless, we will show that it is possible to obtain a graphical representation of this set by purely analytic means. Furthermore, we can calculate "from first principles" the infinite number of scales associated with this approximately self-similar object, which enable us to obtain the generalized fractal dimensions $D(q)$, and the spectrum of scaling indices $f(\alpha)$ by applying the formalism presented in Refs. 1 and 2. Our approach is not restricted to the special Julia set discussed in the following.

Let $F(z)=z^3-1$, $z\in C$. The associated Newtonian map is

$$N(z)=z-F(z)/F'(z)=\tfrac{2}{3}z+1/3z^2. \tag{1}$$

Its attractive fixed points are the roots of F, namely,

$$z_k^*=e^{i2\pi k/3},\quad k=0,1,2. \tag{2}$$

The Julia set J of N is the *simultaneous* boundary of the basins of attraction of the z_k^*, and can be obtained also as the closure of the inverse orbit I of the repelling fixed point ∞ of N (Refs. 3 and 12): $J=\bar{I}$, where

$$I=\{z\in C\,|\,N^l(z)=0;\ l=0,1,2,\ldots\}\,. \tag{3}$$

Let us first show how the structure of I can be completely understood without computer assistance. We define $I_0=\{0\}$, $I_n=\{z\in C\,|\,N^n(z)=0\}$, $n\in N$, implying that $I=\cup_{n=0}^{\infty}I_n$. One immediately finds

$$I_1=\{\hat{z}_0,\hat{z}_1,\hat{z}_2\}=\{2^{-1/3}e^{i(2k+3)\pi/3}\,|\,k=0,1,2\}\,. \tag{4}$$

Now $M_0\equiv(-\infty,\hat{z}_0]$ is mapped on $R_-\equiv(-\infty,0]$ by N. Therefore each I_n will have precisely one element on R_-. Let us call these ponts x_n.

Solving the cubic equations involved we can generate the x_n by *forward iteration,*

$$x_{n+1}=g(x_n)=g^{(n)}(x_1),\quad n\in N\,, \tag{5a}$$

where

$$g(x)=x\{\tfrac{1}{2}+\cosh[\tfrac{1}{3}\cosh^{-1}(1-2x^{-3})]\}\,. \tag{5b}$$

Two additional infinite subsets of I are readily found by determining the other preimages $M_\pm$ of the manifold R_-. We obtain

$$M_\pm=\{\zeta=\xi+i\eta\in C\,|\,\eta=\pm(\xi^{1/2}-\xi^2)^{1/2};\ 0\le\xi\le 2^{-4/3}\}\,. \tag{6}$$

For $n\in N$ we denote by $\zeta_n\equiv\xi_n+i\eta_n$ the unique point of I_n on the arc M_+. Solving again cubic equations one easily gets

$$\xi_{n+1}=h(x_n)=h\circ g^{(n-1)}(x_1)\,, \tag{7a}$$

where

$$h(x)=(-x)\sinh^2\{\tfrac{1}{3}\sinh^{-1}[(-x)^{-3/2}]\}\,. \tag{7b}$$

Altogether we have

$$\zeta_1=\hat{z}_2\,,$$

$$\zeta_{n+1}=h\circ g^{(n-1)}(x_1)+i\{[h\circ g^{(n-1)}(x_1)]^{1/2}+[h\circ g^{(n-1)}(x_1)]^2\}^{1/2}\,. \tag{8}$$

Now $N(z)$ is not only invariant under complex conjugation, which is interchanging $M_\pm$, but also under the rotation $D(z)=e^{i2\pi/3}z$. Thus there are points of I equivalent to x_n, ζ_l, and $\bar{\zeta}_l$ on the six manifolds $D(M_0)$, $D^2(M_0)$, $D(M_\pm)$, and $D^2(M_\pm)$, as shown in Fig. 1.

It is then clear how this scheme for generating the set I has to be iterated: The nine manifolds in Fig. 1 represent the first member E_1 of a sequence of smooth partial embeddings. Each of the 27 manifold preimages of E_1 will support an additional infinite subset of I, and so on. Note that the end points of the arcs forming $E_{n+1}=N^{-1}(E_n)$ are already determined by well-defined points on the manifolds collected in E_n. In the limit $n\to\infty$, $\cup_{k=1}^{n}E_n$ provides us with a complete embedding of I, whose elements emerge as vertices: At the points constituting I_n, $n\ge 2$, precisely ten manifolds touch, namely, one from E_{n-1}, three from E_n, and six from E_{n+1}.

As an example, the result of four steps of manifold preimaging is presented in Fig. 2.

Our construction is very efficient: Each single operation contributes a whole class of points to I, including especially preimages of 0 of arbitrary high order, which are not accessible to ordinary inverse iteration. In addition, the embeddings E_n rapidly trace out the detailed skeleton of J and completely reveal the organizational principle of this infinitely nested object (compare, e.g., our Fig. 2 with the computer-generated pictures of J in Refs. 3, 13, 15, and 16).

The resulting structure is evidently self-similar, at

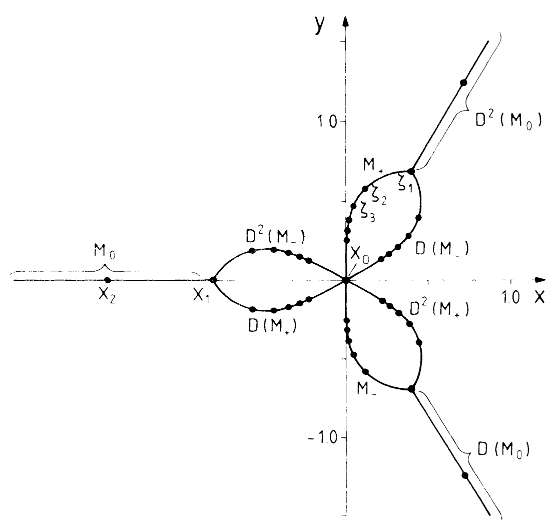

FIG. 1. Manifold preimages of the first generation, and some of the points of I accommodated by them.

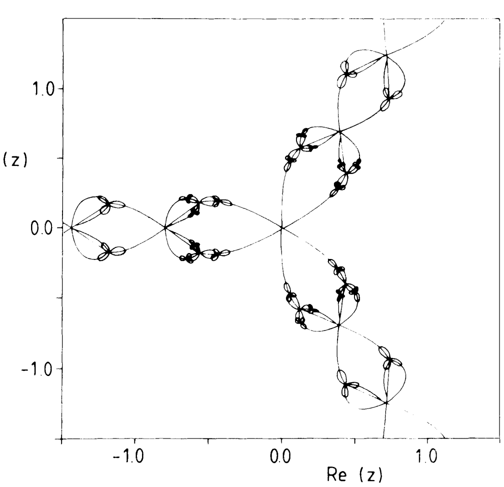

FIG. 2. Union of E_1, E_2, E_3, and E_4.

least in an approximate fashion. This is an almost trivial consequence of the properties of the generating map N, which is conformal and locally contracting. There are, however, various different ways of resolving the whole object into nearly congruent parts. The best one regarding quantitative analysis is suggested by the alternative skeleton construction sketched in Fig. 3, an obvious extension of the principles already employed. Because of symmetry and self-similarity we may focus on the left leaf Λ of the central flower of J, whose points satisfy $-2^{-1/3} \le \mathrm{Re}(z) \le 0$.

The structure emerging from this scheme may be called a "fractal rosary": The original leaf Λ consists of an infinite number of reduced copies, which come in pairs and form two strings along $D^2(M_-)$- and $D(M_+)$-like beads on a rosary. Therefore, an infinity of different and independent scales are involved and, in a first approximation, these scales are specified analytically by the points ζ_n.

Based on these observations, a quantitative description of the Julia set in terms of multifractal calculus[1,2] is readily achieved. We keep on restricting attention to the compact set Λ, as the central blossom contains $\frac{2}{3}$ of all the points in I and the weight of the infinite object would scale with the "radial dimension" zero for any reasonable mass distribution. The key to our analysis is the Hentschel-Procaccia scheme for renormalizable multifractals; it is built on the assumption that the strange set considered may be resolved into a collection of disjoint down-scaled copies of the original, which in turn can be resolved in precisely the same way, etc.[1] A generalization of this scheme operating with an infinity of distinct scales is applicable to our problem and is implemented by determining the renormalization parameters $s_m \equiv l_m^{-1}$, M_m, and p_m, $m \in N$. These parameters have a simple meaning: m denumerates the different classes of copies consisting of M_m identical members, which are reduced in size by the factor s_m as compared to the original. p_m is the probability or weight of *one* element in the mth class with respect to a properly defined measure.

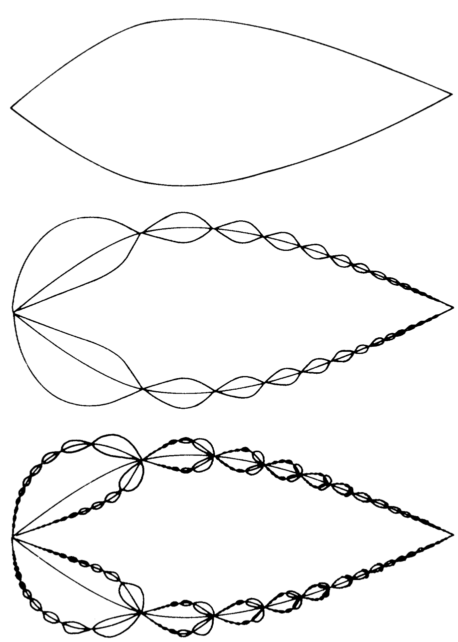

FIG. 3. Schematic representation of the first three steps of a skeleton construction involving an infinity of manifolds preimages at each step.

As the reduced copies Λ_m of Λ come in pairs we have $M_m = 2$. Within our first approximation we define the scaling parameters l_m as the normalized arc lengths on M_+ between ζ_m and ζ_{m+1}, i.e.,

$$l_m = 2^{1/3} \int_{\zeta_{m+1}}^{\zeta_m} d\xi \left[\frac{\frac{1}{2}\xi^{1/2} + 1/16\xi}{\xi^{1/2} - \xi^2} \right]^{1/2} . \tag{9}$$

Using (7) we find that

$$l_m \underset{m \to \infty}{\longrightarrow} c_\infty \tfrac{2}{3}^{m/2}, \quad c_\infty = 0.1986\ldots . \tag{10}$$

Another obvious choice for the l_m is to take the normalized Euclidean distances $|\zeta_m - \zeta_{m+1}|$. This probably results in a small underestimation of the scales.

A subtle point when dealing with Julia sets is the definition of a reasonable probability measure on J in order to specify the p_m. This measure is dependent on the way in which points of the Julia set are ordered. The "natural measure" μ is the one obtained by distributing a unit mass equally over the points of $I_s \cap \Lambda$ and performing the limit $s \to \infty$. μ is the unique measure with the maximal entropy.[17]

A crucial advantage of our resolution of Λ into pairs of subleaves Λ_m is that we immediately obtain the *exact* result

$$p_m \equiv \mu(\Lambda_m) = 3^{-m}, \quad m \in N , \tag{11}$$

by inspecting the construction process for the E_n.

Now within the Hentschel-Procaccia approach all the generalized fractal dimensions $D(q)$ are determined implicitly or explicitly by very simple formulas[1]:

$$\sum_m M_m p_m^q l_m^{(1-q)D(q)} = 1, \quad q \neq 1 ; \tag{12}$$

and

$$D(1) = \sum_m M_m p_m \ln p_m \Big/ \sum_m M_m p_m \ln l_m . \tag{13}$$

Equation (12) may be even further simplified by calculating only the first ten or twenty scales l_m via (9) and summing up the geometric tail using the result (10). Thus all $D(q)$ are found by elementary calculus. For the most interesting entity, the similarity dimension, we obtain $D(0) = 1.429\ldots$ using (9). This is in excellent agreement with a brute-force numerical test on the basis of a million points of I, which yields $D(0) = 1.42$. From $D(q)$ the complete spectrum $f(\alpha)$ of scaling indices is derived by Legendre transformation[2] in the usual way; the results will be detailed in a forthcoming paper.[18]

The boundary values $\alpha_{\min}$ and $\alpha_{\max}$ of the support of f are directly calculable:

$$\alpha_{\min} = D(\infty) = \inf_m (\ln p_m / \ln l_m) = \ln p_1 / \ln l_1 = 1.146\ldots,$$

$$\alpha_{\max} = D(-\infty) = \sup_m (\ln p_m / \ln l_m) = \lim_{m \to \infty} \ln p_m / \ln l_m = 2\ln 3 / \ln \tfrac{3}{2} = 5.419\ldots.$$

A rather different measure $\tilde{\mu}$ on J is suggested by the alternative construction shown in Fig. 3, which drastically rearranges the "time order" for the generation of the points in I. The same weight may be assigned to each Λ_m, as there is a one-to-one correspondence between those points of I they accommodate. Taking some care of the limit processes involved the production of a well-defined $f(\alpha)$, whose support is $[0,\infty)$.[18] Other multifractals, whose Hölder exponent α ranges all the way from 0 to ∞ have been discussed recently by Gutzwiller and Mandelbrot[19] (see also Ref. 20).

In conclusion we have demonstrated that a "paper and pencil construction" of a complicated Julia set representing the simultaneous boundary of three distinct basins of attraction is indeed possible. Based on this construction, already the simplest analytic approximation gives excellent quantitative results for the complete family of generalized fractal dimensions. In fact, our scheme could be made as precise as desired if the l_m are determined from higher-order manifold preimages and if the scales converge uniformly within Λ under backward iteration. This question will be discussed elsewhere.[18] Working out for the first time analytically the structure of a self-similar Julia set also clarifies the applicability in this case of the recently proposed theory of generalized fractal dimensions.[1,2]

Quite contrary to the usual situation faced in nonlinear science, the difficulty for the problem considered here lies in the numerical verification of the analytical assertions.[5] Note that our methods work best for $q \to -\infty$ where computer tests are unfeasible, because the calculator would have to run virtually forever to explore the sparsest regions of I.

After completion of this work we learned about a paper by Procaccia and Zeitak,[21] where an analytic approach to invariant measures associated with connected Julia sets emerging from polynomial mappings is discussed. The authors make use of exact encodings and the transfer-matrix formalism developed in Ref. 22. Their beautiful idea (independently advanced also by Bohr, Cvitanović, and Jensen[23]) of representing fractal diffusion-limited aggregates by special Julia sets suggests that the investigation of strange repellers becomes increasingly relevant to nonlinear physics.

We thank J. Deutsch and D. Saupe for useful discussions. The help of G. Börger, A. Block, and W. v. Bloh concerning numerical work is gratefully acknowledged. This work was supported in part by the Deutsche Forschungsgemeinschaft. One of us (H.J.S.) acknowledges financial support from the Heisenberg Fellowship.

[a]Permanent address: Fachbereich Physik, Universität Oldenburg, D-2900 Oldenburg, FRG.

[1]H. G. E. Hentschel and I. Procaccia, Physica (Amsterdam) **8D**, 435 (1983).

[2]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and P. E. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[3]P. Blanchard, Bull. Amer. Math. Soc. **11**, 1 (1984).

[4]One exception is the classic work of D. Ruelle, Ergodic Theory Dynamical Systems **2**, 99 (1982).

[5]D. Saupe, Physica (Amsterdam) **28D**, 358 (1987).

[6]See, e.g., S. Marmi, J. Phys. A **21**, L961 (1988).

[7]N. Manton and M. Nauenberg, Commun. Math. Phys. **89**, 555 (1983).

[8]L. P. Kadanoff, Proc. Nat. Acad. Sci. U.S.A. **81**, 1276 (1984).

[9]M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983); S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, *ibid.* **50**, 1873 (1983).

[10]See, e.g., R. Rammal, J. Phys. **45**, 191 (1984).

[11]A. Cayley, Quart. J. Pure Appl. Math. **16**, 179 (1879).

[12]J. Hubbard (unpublished); see the essay by A. Douady, in *The Beauty of Fractals* (Springer-Verlag, Berlin, 1986).

[13]J. P. Eckmann, Recherche **14**, 260 (1983).

[14]J. H. Curry, L. Garnett, and D. Sullivan, Commun. Math. Phys. **91**, 267 (1983).

[15]H. O. Peitgen, D. Saupe, and F. v. Haeseler, Math. Intelligencer **6**, 11 (1984).

[16]See H. O. Peitgen and P. H. Richter, *The Beauty of Fractals* (Springer-Verlag, Berlin, 1986), and references therein.

[17]See A. Manning, Ann. Math. **119**, 425 (1984), and references therein.

[18]M. Nauenberg, H. J. Schellnhuber, and G. Börger (unpublished).

[19]M. C. Gutzwiller and B. B. Mandelbrot, Phys. Rev. Lett. **60**, 673 (1987).

[20]B. B. Mandelbrot (to be published).

[21]I. Procaccia and R. Zeitak, Phys. Rev. Lett. **60**, 2511 (1988).

[22]M. J. Feigenbaum, M. H. Jensen, and I. Procaccia, Phys. Rev. Lett. **57**, 1503 (1986); M. H. Jensen, L. P. Kadanoff, and I. Procaccia, Phys. Rev. A **36**, 1409 (1987).

[23]T. Bohr, P. Cvitanović, and M. H. Jensen, Europhys. Lett. **6**, 445 (1988).