Defect-Mediated Turbulence

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We describe a turbulent state characterized by the presence of topological defects. This "topological turbulence" is likely to be experimentally observed in nonequilibrium systems.

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The problem of phase transitions received a great deal of attention during the 1970's.[1] Especially exciting has been the study of transitions in 2D systems.[2] The specificity of such transitions is related to the predominant role of defects.[3] In a quite different context, the destruction of the macroscopic two-dimensional order presents striking similarities[4] with these transitions. Theoretical attempts[5] to explain these similarities take into account the presence of stochastic fluctuations. The aim of this Letter is to reconsider this problem in the light of ideas coming from dynamical systems theory.[6] More precisely, using numerical simulations performed on a deterministic dynamical model, we intend to clarify a mechanism of defect creation in macroscopic structures. Our main result is that phase turbulence[7] in sufficiently extended bidimensional systems leads to the creation of topological defects.[8] The resulting dynamical state has been studied in detail by means of numerical simulations performed on the simplest model which displays this behavior.

Among all the transitions in macroscopic systems, the simplest one consists of the appearance of spatially synchronized oscillations. Such transitions to "temporal order" arise naturally in chemical and biological contexts.[7,9,10] We consider an isotropic, two-dimensional medium, and assume that it starts oscillating when some control parameter is varied. The typical quantity Q describing the system, as a chemical concentration, reads

$$Q = Q_0 + \{A\exp(i\omega_0 t) + \bar{A}\exp(-i\omega_0 t)\} + \cdots .$$

The first term in this expansion describes the constant, time-translationally invariant part of Q, while the second one accounts for the symmetry breaking; the ellipses stand for small corrections. Such an expansion is only valid when $|A|$, $|\partial_t A|$, and $|\nabla A|$ are small enough. The complex field A, which measures the "amount of broken symmetry,"[11] is called the order parameter. It obeys a Ginzburg-Landau[12] type equation:

$$\partial A/\partial t = (\mu_r + i\mu_i)A + (\alpha_r + i\alpha_i)\nabla^2 A - (\beta_r + i\beta_i)|A|^2 A , \quad (1)$$

where μ_i represents a shift in frequency, μ_r measures the deviation from the transition threshold, α_i and β_i are associated with dispersion effects and nonlinear frequency renormalization, and α_r and β_r correspond to diffusion effects ($\alpha_r > 0$) and to nonlinear saturation ($\beta_r > 0$). In the spatially homogeneous case, Eq. (1) is the amplitude equation of the so-called Hopf bifurcation.[13] When all the coefficients are real numbers, one recovers the mean-field theory (Ginzburg-Landau) of superfluidity. In this limit, Eq. (1) is the potential, and if thermal noise is added, the dynamics eventually converge toward the defect-free solution, the absolute minimum of the Ginzburg-Landau potential. In the following, we consider the deterministic evolution associated with Eq. (1), and we will show that it contains, for some range of parameters, a mechanism of defect creation.

The so-called perfect temporal pattern corresponds to a particularly simple solution of Eq. (1),

$$A_0 = \left(\frac{\mu_r}{\beta_r}\right)^{1/2} \exp\left[i\left(\mu_i - \frac{\beta_i}{\beta_r}\mu_r\right)t\right].$$

When $\alpha_r + \alpha_i\beta_i/\beta_r > 0$, such a solution is linearly stable. Sideband or modulational instability[6] arises when $\alpha_r + \alpha_i\beta_i/\beta_r$ changes sign. Near the instability threshold, one can reduce the dynamics of Eq. (1) to an equation for the phase only,

$$\partial_t \phi = (\alpha_r + \alpha_i\beta_i/\beta_r)\nabla^2\phi + \cdots , \quad (2)$$

where ϕ is implicitly defined, at leading order, by

$$A = \left(\frac{\mu_r}{\beta_r}\right)^{1/2}\left[1 - \frac{\alpha_i}{2\mu_r}\nabla^2\phi - \frac{\alpha_r}{2\mu_r}(\nabla\phi)^2 + \cdots\right] \times \exp\left[i\left(\mu_i - \frac{\beta_i}{\beta_r}\mu_r\right)t + i\phi\right],$$

and A obeys Eq. (1). The nonlinear extension of Eq. (2)[7,14] gives rise to turbulent regimes,[15,16] which correspond to "phase turbulence"[7] (see Fig. 2, $\beta_i = -0.80$). This turbulence leads to a weak destruction of the temporal order induced by the periodic pattern, since the correlation function of the field $\langle A(x_1,y_1,t)\bar{A}(x_2,y_2,t)\rangle$, where $\langle\ \rangle$ denotes time averaging, decreases very slowly. Far from the phase instability threshold ($\alpha_r + \alpha_i\beta_i/\beta_r \ll -1$), the adiabatic elimination of the amplitude mode which leads to Eq. (2) breaks down. The turbulent-state solution of Eq. (1) which strongly couples amplitude and phase modes has been termed as "amplitude tur-

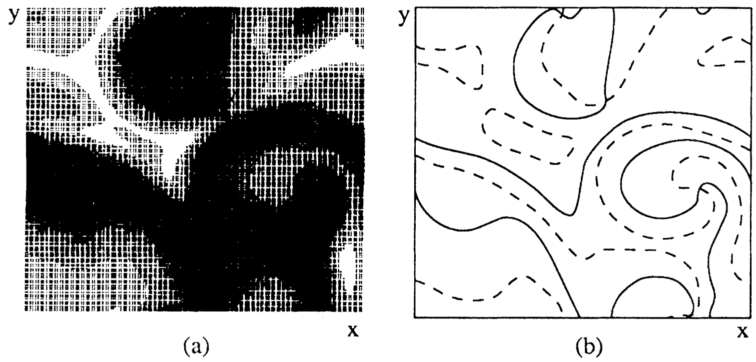

FIG. 1. (a) Instantaneous state of Re(A) ($\alpha_i=2$, $\beta_i=-0.85$). (b) The corresponding lines Re(A)=0 (solid) and Im(A)=0 (dashed).

bulence."[7,17] This turbulence leads to a strong destruction of the temporal order characterized by an exponential decrease of correlations (see Fig. 2, $\beta_i=-3.00$). When μ_r, α_r, and β_r are small, Eq. (1) can be seen as a perturbation of the 2D nonlinear Schrödinger equation. This leads[17] to an interpretation of the strong amplitude turbulence in terms of the self-focusing phenomenon.[6] The aim of this Letter is to underscore the existence of a turbulent regime associated

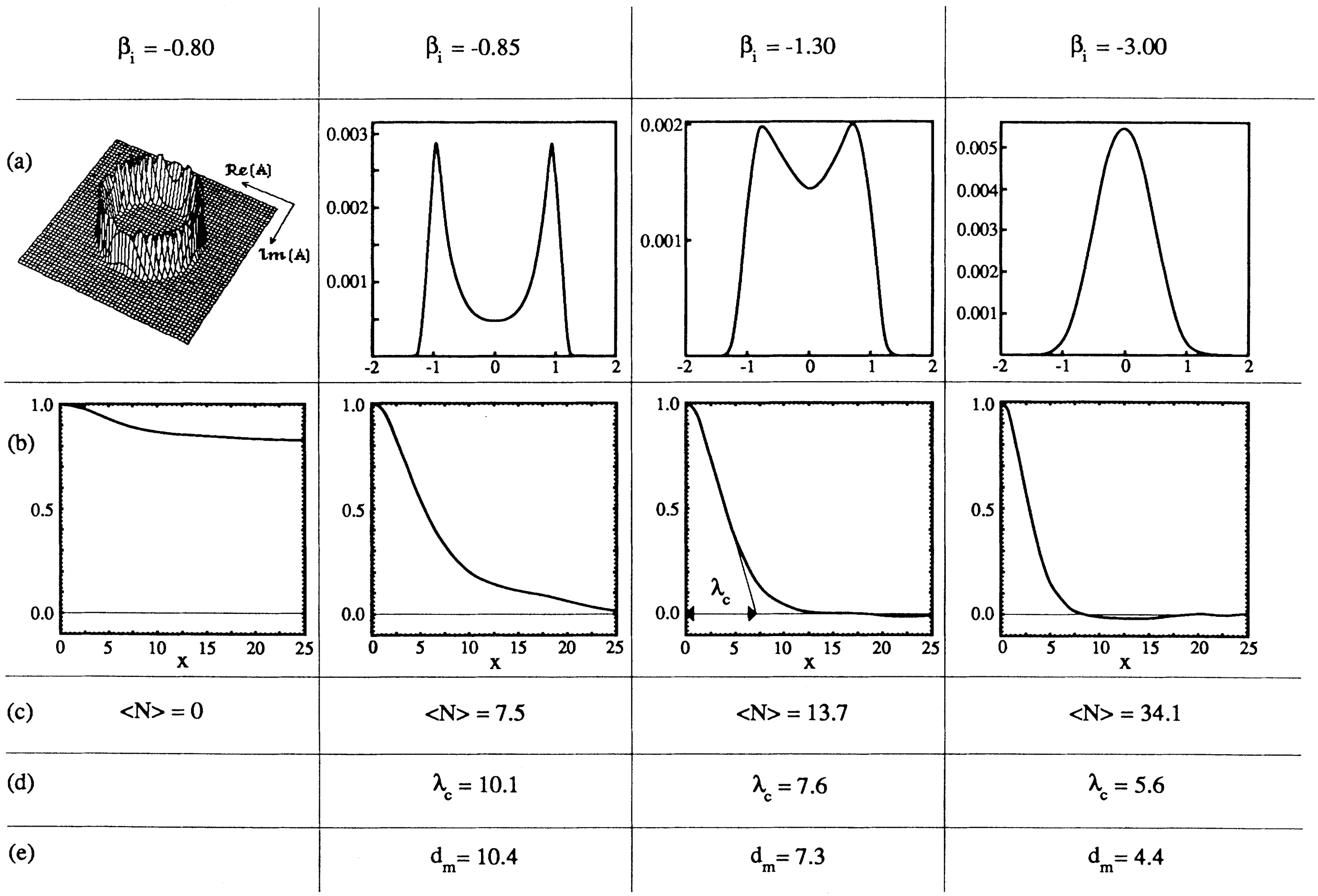

FIG. 2. Numerical results for $\alpha_i=2$ and various values of β_i. (a) Distribution of the complex field A $\{\rho(a+ib)da\,db = P[a \le \mathrm{Re}(A) < a+da$ and $b \le \mathrm{Im}(A) < b+db]\}$ computed by time averaging for $\beta_i=-0.8$, and its successive radial cross sections. (b) Correlation function as a function of position. (c) Mean number of defects in the box. (d) Correlation length. (e) Mean distance between a defect and its nearest neighbor.

with the presence of topological defects, which turns out to interpolate between the two forms of turbulence described above (see Fig. 2, $\beta_i = -0.85$ and $\beta_i = -1.30$). Actually, when increasing $|\alpha_r + \alpha_i\beta_i/\beta_r|$, amplitude modes are progressively awakened and, in two dimensions, the existence of localized amplitude modes (stable topological defects) allows a nucleation transition, which is reminiscent of those observed in condensed matter physics.[18]

Stable topological defects[8] are singular solutions of Eq. (1). At the core of a defect, the real part and the imaginary part of the order parameter A both vanish. This solution takes the form of a spiral wave[9] (see Fig. 1) which propagates out of the core. In what follows, we place ourselves in the phase-unstable regime, namely, $\alpha_r + \alpha_i\beta_i/\beta_r < 0$, and study the behavior of the solutions of Eq. (1). Initial conditions are chosen close enough to the homogeneous solution A_0: $\mu_r = 1$, $\mu_i = 0$, $\alpha_r = \beta_r = 1$.[19] The simulations reported here have been performed on a CRAY-2 with a 2D periodic-boundary-condition spectral code, with 80×80 collocation points, in a box of 50×50 units of length.

As expected, the system first follows a phase regime. Then, the dynamics eventually evolve toward a more disordered regime due to the nucleation of pairs of defects. The defect creation mechanism is simple enough. From time to time, a phase gradient becomes important. This leads to a pinching of the equiphases, and eventually to a shocklike event which, in turn, is responsible for the creation of the pair.[20] Figure 1(a) displays, in the asymptotic regime, an instantaneous state of the real part of A which exhibits a small number of spiral defects [see the equiphases in Fig. 1(b)]. Large areas with quasicoherent phase [black and white regions in Fig. 1(a)], separated by the arms of the spiral waves, are observed. In the time evolution, defects move, new pairs are created, and others annihilate. This leads to a complex spatio-temporal behavior termed "topological turbulence," which is characterized by short-range correlations (see Fig. 2, $\beta_i = -0.85$ and $\beta_i = -1.30$). In this regime, the correlation length is found to be of the same order as the mean distance between a defect and its nearest neighbor (see Fig. 2), which supports the idea of "defect-mediated turbulence."

Various numerical simulations have been performed, for different values of α_i and β_i. Figure 3 displays the mean number of defects[21] in the box as a function of α_i, for $\beta_i = -2$. We have checked numerically that this number scales linearly with the size of the box. The behavior observed in Fig. 3 suggests a first-order phase transition, which is confirmed by the existence of a hysteresis loop.

For α_i fixed, the mean number of defects decreases when $|\beta_i|$ decreases. For small values of $|\beta_i|$, the number of defects as a function of time is strongly intermittent and alternates between zero and a mean number

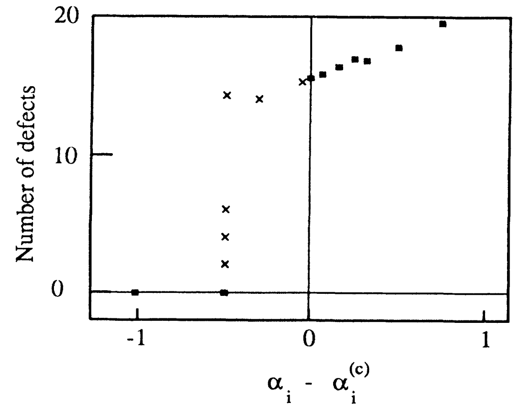

FIG. 3. Number of defects as a function of $\alpha_i - \alpha_i^{(c)}$ for $\beta_i = -2$ ($\alpha_i^{(c)} = -1/\beta_i = \frac{1}{2}$). Squares are obtained with initial conditions chosen around the homogeneous solution (see text). Crosses correspond to hysteresis.

of defects which decreases with $|\beta_i|$. As $|\beta_i|$ increases, more and more delocalized amplitude modes are awakened, and topological turbulence transforms itself progressively into amplitude turbulence (see Fig. 2).

Topological turbulence exists in other physical systems, as for instance in those giving rise to wave patterns.[22] The defects associated with these patterns have been discussed in a previous work.[23] They are dislocations and domain walls from which two counterpropagating waves emanate. As expected, the topological disorganization of such patterns exhibits these two kinds of defects when an external constraint is varied.[24,25] Work in this direction is in progress.

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