Chiral Quantum Baryon

Juan A. Mignaco
Centro Brasileiro de Pesquisas Físicas, Conselho Nacional de Desenvolvimento Científico e Tecnológico, 22290 Rio de Janeiro, Brazil

Stenio Wulck
Instituto de Física, Universidade Federal do Rio de Janeiro, CP 68528, 21944 Rio de Janeiro, Brazil
(Received 23 December 1988)

We show that a classical soliton for the nonlinear SU(2) σ model in the hedgehog configuration admits a stable solution, when quantized through collective coordinates, which may be identified with the nucleon. The whole approach depends on a single, dimensional, and arbitrary constant. Numerical results seem to converge for the mass and for the right value of the weak axial-vector coupling.

PACS numbers: 11.10.Lm, 11.30.Rd, 11.40.Fy

It is widely believed after the work of several authors,[1] who revived the argument by Skyrme,[2] that a baryon is a soliton of a chiral theory. Classical stability arguments seemed to require, however, the introduction of an additional term to the nonlinear σ-model Lagrangian (in the nonrelativistic limit)

$$\mathcal{L} = -\tfrac{1}{2} f_\pi^2 \int d^3x \,\mathrm{Tr} \sum_{k=1}^{3} (\partial_k U^\dagger)(\partial_k U) , \tag{1}$$

where U is a unitary operator,

$$UU^\dagger = 1 ,$$

and f_π is the usual pion-decay constant. The additional term introduced by Skyrme,

$$-\frac{1}{32e^2} \int d^3x \,\mathrm{Tr}[U^\dagger(\partial_k U), U^\dagger(\partial_l U)]^2 , \tag{2}$$

incorporated a dimensional parameter e. Several works[3] dealt with the phenomenology of this classically stable theory, and showed, after quantization, a reasonable agreement for physical quantities when the hedgehog form for U was used (spherically symmetric *Ansatz*):

$$U_0 = \exp[i\boldsymbol{\tau}\cdot\mathbf{n}F(r)] , \tag{3}$$

where τ_k represent the usual Pauli matrices for SU(2) and

$$\mathbf{n} = \mathbf{r}/|\mathbf{r}| , \tag{4}$$

$$r = \sum_{k=1}^{3} (x^k)^2 . \tag{5}$$

There are several points which deserve further attention. First, it is usually assumed that the effective chiral Lagrangian should result from some more fundamental theory, for instance, from a gauge theory such as QCD, and it is not easy to see how to generate from it a term like (2). Second, it is not obvious how to ascribe a physical meaning to the new dimensional constant in the game, e. Some recent work attempts to relate it to the pion-decay constant f_π.[4] Third, using the full Skyrme Lagrangian leads to numerically encouraging results, but the formal results for the description of chiral dynamics at low energies do not seem to depend on e.[5]

Lately, we have addressed the question of the meaning of a theory without a Skyrme term.[6] In particular, we have stressed the point that the classical Euler-Lagrange equation for $F(r)$ is singular and introduces a dimensional constant in the formalism. This constant carries, in the classical domain, the instability of the nonlinear classical σ-model soliton. It seems that former work overlooked this constant. In fact, some feeling about it is present in the work by Balachandran,[7] who introduced a kind of variational "shape" parameter, accounting for the size of the soliton.

As we showed in Ref. 6, this constant appears naturally when one sets out to solve the classical equation of motion for the Lagrangian (1) using the hedgehog SU(2) solution (3):

$$d^2F(r)/dr^2 + (2/r)[dF(r)/dr] = \sin 2F(r) . \tag{6}$$

To eliminate the first derivative, one uses

$$F(r) = \chi(r)/r \tag{7}$$

and, calling

$$r = 2x , \tag{8}$$

we arrive finally at

$$d^2\chi(x)/dx^2 = (2/x)\sin[\chi(x)/x] . \tag{9}$$

It is easy to verify that for the second derivative we arrive at an identity, and so it remains a free, dimensional, parameter. In order to solve (9), we must require, for consistency of both sides,

$$\chi(0) = 0 , \tag{10}$$

$$\chi'(0) = 0, \pm 2n\pi , \quad n = 1,2,\ldots . \tag{11}$$

To have a soliton solution with winding number n,

$$F(0) = -n\pi , \tag{12}$$

$$\chi'(0) = -2n\pi , \tag{13}$$

provided $F(r)$ is zero at infinity, and we have at the end

$$\chi(x) = -2n\pi x + \tfrac{1}{2}\chi''(0)x^2 X([\chi''(0)x]^2) , \tag{14}$$

where

$$X(s)=\sum_{n=1}^{\infty} f_n s^{2(n-1)}\,, \tag{15}$$

$$\chi''(0)=d^2\chi(x)/dx^2|_{x=0}\,, \tag{16}$$

and $s=\chi''(0)x$ is a dimensionless variable. The first coefficients in the expansion of $X(s)$ are

$$f_1=1\,,\quad f_2=-\frac{1}{5!}=-\frac{1}{120}$$

$$f_3=\frac{1}{6!}\frac{3^2}{2^4\times 7}=\frac{1}{8960}$$

$$f_4=-\frac{1}{8!}\frac{17}{2^4\times 3\times 5}=-\frac{17}{9\,676\,800}\,,$$

$$f_5=\frac{1}{10!}\frac{3\times 7\times 73}{2^8\times 5\times 11}=\frac{73}{2\,433\,024\,000}\,,$$

$$f_6=-\frac{3337}{6\,199\,345\,152\,000}\,.$$

The appearance of the dimensional parameter $\chi''(0)$ for the solution of the soliton has not been noticed by the authors of previous work. It seems, however, as we mentioned earlier, that Balachandran and co-workers[1,7] were somewhat aware of its necessity when they introduced a variational *ad hoc* shape parameter. Besides, notice that this parameter should even be included with the Skyrme term [Eq. (2)], since it does not contribute to the singularity at the origin.

It turns out that the chiral angle itself, $F(r)$, is in fact a function of the dimensionless variable s, as seen replacing (14) in (7):

$$F(r)=F(s)=-n\pi+\tfrac{1}{4}sX(s)\,. \tag{17}$$

This new dimensional parameter, which, we stress, comes from the consistency of the series solution at the origin for the chiral angle, is intimately connected to the usual stability argument against the soliton solution for the nonlinear σ-model Lagrangian. If we write the expression for the mass of the soliton,

$$M_0=4\pi f_\pi^2\int_0^\infty dr'[r'^2(dF/dr')^2+2\sin^2F(r')]\,, \tag{18}$$

in terms of Eq. (17) above, we find

$$M_0=2\pi f_\pi^2\frac{1}{\chi''(0)}\int_0^\infty ds'\{\tfrac{1}{4}s'^2\mathcal{F}'^2(s')+8\sin^2[\tfrac{1}{4}\mathcal{F}(s')]\}\,, \tag{19}$$

putting

$$\mathcal{F}(s)=sX(s) \tag{20}$$

with $\mathcal{F}'(s)$ being its first derivative. The integral over the dimensionless variable s' in Eq. (19) is a pure number, and the usual argument for the instability of the soliton, coming from the replacement

$$r'\to\lambda r'$$

in Eq. (18), translates into the instability under a variation of $\chi''(0)$.

It is well known, though, that when quantizing with the help of collective coordinates

$$U(\mathbf{r},t)=A(t)U_0(\mathbf{r})A^\dagger(t)$$
$$=\cos F(r)+i\tau_j D_{jk}(t)n_k\sin F(r)\,, \tag{21}$$

where $D_{jk}(t)$ are rotation matrices, the expression for the energy of the quantized system becomes the one for a rotating top (see, for instance, Balachandran[7] or Adkins, Nappi, and Witten[3]),

$$M=M_0+(2\lambda)^{-1}\mathbf{J}^2\,, \tag{22}$$

where the "momentum of inertia," λ, is

$$\lambda=\tfrac{16}{3}f_\pi^2\int_0^\infty dr'r'^2\sin^2F(r')\,. \tag{23}$$

Using Eq. (17),

$$\lambda=2\pi f_\pi^2\frac{1}{\chi''(0)^3}\int_0^\infty ds'\,\tfrac{64}{3}s'^2\sin^2[\tfrac{1}{4}\mathcal{F}(s')]\,. \tag{24}$$

With this, Eq. (22) takes the form

$$M=[2\pi f_\pi^2/\chi''(0)]a+\tfrac{1}{2}[\chi''(0)^3/2\pi f_\pi^2 b]\mathbf{J}^2\,. \tag{25}$$

The quantization for the symmetric top as a fermion shows that the possible values for $\mathbf{J}^2$ (and for the isotopic spin $\mathbf{T}^2=\mathbf{J}^2$) are half-integer.

It is easily seen that Eq. (25) has a minimum in terms of $\chi''(0)$. The only remaining fixed scale parameter in Eq. (25) is f_π, the pion-decay constant. The values for $\chi''(0)$ and the mass at the minimum are

$$\chi''(0)=\left[\frac{2}{3}\frac{(2\pi)^2}{\mathbf{J}^2}ab\right]^{1/4}f_\pi\,, \tag{26}$$

$$M=\frac{4}{3}\left[\frac{3}{2}(2\pi)^2\mathbf{J}^2\frac{a^3}{b}\right]^{1/4}f_\pi\,. \tag{27}$$

We have immediately a prediction for the mass ratio of the lowest states:

$$M(J=\tfrac{3}{2})/M(J=\tfrac{1}{2})=5^{1/4}\simeq 1.495\ldots\,, \tag{28}$$

which agrees rather well with the known experimental ratio for the Δ resonance and the nucleon:

$$M(\Delta)/M(N)\simeq 1.32\ldots\,. \tag{29}$$

It may seem that we have lost any trace of the value of the "baryon number," or winding number, as it appears in the first term of Eq. (17). This is not the case, since asymptotically the expression for $X(s)$ is well determined.

In order to see this, let us go back to the solution for the chiral angle at infinity, looking for the solution of Eq. (6). Introducing $y=1/x$, using Eqs. (7) and (8), and defining

$$\chi(x)=\psi(y)\,,\quad K(y)=y\psi(y)\,,$$

we arrive readily at

$$K''(y) = (2/y^2)\sin K(y)\,, \tag{30}$$

with the relation

$$F(x) = \tfrac{1}{2} K(y)\,. \tag{31}$$

The series solution of Eq. (30) gives

$$K(y) = 2n_\infty \pi + \tfrac{1}{2} K''(0) y^2 Y(y)\,, \tag{32}$$

with

$$Y(y) = \sum_{j=1}^{\infty} k_j [K''(0) y^2]^{2(j-1)}\,,$$

$$K''(0) = d^2K(y)/dy^2|_{y=0}\,,$$

$$k_1 = 1\,, \quad k_2 = -\frac{1}{6!}\frac{15}{7} = -\frac{1}{336}\,,$$

$$k_2 = \frac{1}{11!} 2\times 3^4 \times 5 = \frac{1}{9280}\,, \quad k_3 = -\frac{1}{6\,209\,280}\,.$$

The winding number of the soliton is given by the difference

$$N = n - n_\infty\,,$$

and so, if $n=1$, in order to have $N=1$, n_∞ must be zero. The dimensional parameter $\chi''(0)$ translates at infinity to the dimensional parameter $K''(0)$ $[\sim -\chi''(0)^{-2}]$. Then, as the radial coordinate grows to infinity,

$$F \sim -K''(0)/r^2\,. \tag{33}$$

Comparing Eq. (33) with Eq. (17), we see that at infinity

$$X(s) \sim +4\pi n/s + O(s^{-2})\,. \tag{34}$$

The behavior at infinity resulting from Eq. (32) allows one to have information about the axial-vector current coefficient g_A, as shown by Adkins, Nappi, and Witten,[3]

$$g_A = \tfrac{8}{3}\, 2\pi f_\pi^2 K''(0)\,. \tag{35}$$

We have begun to work out numerical results for the SU(2) chiral theory. They are at the moment not complete, but we think they deserve some consideration.

In order to exploit our knowledge of the solutions by power-series expansion of Eqs. (9) and (30), we propose a systematic approximation using Padé approximants.[8] They are in this case of a particular type, since we need to enforce the conditions that fix the soliton solution to be of winding number one. Defining

$$f[N,M](\alpha) = \frac{n_1 + n_2\alpha + n_3\alpha^2 + \cdots + n_M \alpha^{M-1}}{1 + d_1\alpha + d_2\alpha^2 + \cdots + d_N\alpha^N}\,, \tag{36}$$

we find that the only approximants satisfying the conditions

$$X[N,M](0) = 1\,, \quad N[N,M](\infty) \sim 4\pi/s + O(s^{-2})\,,$$

are those with $N = 2j+1$, $M = 2j$, $j = 1, 2, \ldots$, i.e., [3,2], [5,4], [7,6], ... For instance, [3,2] for $X(s)$ uses the first coefficient only, and is particularly simple:

$$X[3,2](s) = (1 + s/4\pi)[1 + s/4\pi + (s/4\pi)^2]^{-1}\,. \tag{37}$$

After determining the coefficients in the Padé approximant (36), we calculate the integrals a and b in (25) and find the values of $\chi''(0)$ and M. To have the axial-vector coupling, we use the fact that the asymptotic form for the Padé approximants is

$$F(\sigma) \sim \pi c_1[N]\sigma^2 \quad (\sigma \sim 0)\,, \tag{38}$$

with $\sigma = 1/s$ and

$$c_1[N] = -\pi(d_{N-2}/d_N - n_{N-2}/4\pi d_N)\,. \tag{39}$$

The first results are given in Table I.

We see that the above results show a systematic trend, and further work is currently being done, increasing the order of the approximants (that is, using more information about the soliton solution) and enlarging the flavor group. The dimensional parameter $\chi''(0)$ is rather large, showing the importance of short-distance behavior. The value for the mass is rather low, and seems to converge to a value around 0.50 GeV for our chosen value for f_π. Interestingly, the results for the axial-vector weak coupling look nice, and may converge to the right value.

We think that the above results indicate that the dynamical information available from low-energy hadronic physics summarized in the current-algebra effective Lagrangian given by Eq. (1) provides already a consistent framework for description of the nucleon and the lower baryon states after quantization. The need to use a minimum of the quantum energy for a description of baryons does not seem to be too extravagant. It arises from the exact behavior of the hedgehog classical solution.

If, on the other hand, one expects to describe low-energy hadron physics from a dynamical quantum theory such as QCD through an effective Lagrangian, experience with two dimensions[9] seems to indicate that quantum (loop) effects are relevant.

One may also recall that the simple hydrogen atom is classically unstable, and the crudest quantum condition makes it into a stable, quantized system. The comparison may look exagerated, but it is worth remembering that the quantum system does not always follow the paths suggested by classical dynamics.

TABLE I. Results of the first approximations. We have taken $f_\pi = 0.067$ GeV.

	$\chi''(0)/f_\pi$	$\chi''(0)$ (GeV)	M_N/f_π	M_N (GeV)	g_A
[3,2]	193.2	13.00	9.964	0.6676	0.891
[5,4]	371.1	24.87	8.693	0.5824	1.087
[7,6]	624.1	41.82	8.166	0.5471	1.162

The authors wish to thank J. S. Helman, L. J. Mignaco, and J. E. Stephany Ruiz for their help in the computation of numerical results. Useful conversations with Professor A. P. Balachandran, and his interest in this work, are warmly acknowledged.

Note added.— After this work was completed, we became aware of the work of P. Jain, J. Schechter and R. Sorkin [Phys. Rev. D. **39**, 998 (1989)], who agree with our general framework.

[1]N. K. Pak and H. C. Tze, Ann. Phys. (N.Y.) **117**, 164 (1979); A. P. Balachandran, V. P. Nair, S. G. Rajeev, and A. Stern, Phys. Rev. Lett. **49**, 1124 (1982); **50**, 1630(E) (1983); E. Witten, Nucl. Phys. **B223**, 422 (1983); **B223**, 433 (1983).

[2]T. H. R. Skyrme, Proc. Roy. Soc. London A **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).

[3]G. Adkins, C. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983); M. P. Mattis and M. Karliner, Phys. Rev. D **31**, 2833 (1985); M. P. Mattis and M. E. Peskin, Phys. Rev. D **32**, 58 (1985). For further coverage, see the excellent recent review by Ulf G. Meissner, Phys. Rep. **161**, 213 (1988).

[4]H. G. Dosch and S. Narison, Phys. Lett. B **184**, 78 (1987).

[5]See, for instance, the work by H. J. Schnitzer, Phys. Lett. **139B**, 217 (1984).

[6]J. A. Mignaco and S. Wulck, contribution to the volume celebrating the 70th birthday of Prof. J. Leite Lopes, Notas de Física (CBPF) 050/88 (unpublished).

[7]A. P. Balachandran, Syracuse University Report No. SU-4222-314, 1985 (unpublished), lectures delivered at the Theoretical Advanced Study Institute in Elementary Particle Physics, Yale University, 1985.

[8]G. A. Baker, Jr., Adv. Theor. Phys. **1**, 1 (1965).

[9]This may be illustrated by the case of QCD_2, as found in A. J. D'Adda, A. C. Davis, and P. Di Vecchia, Phys. Lett. **121B**, 335 (1983); A. M. Polyakov and P. B. Weigmann, Phys. Lett. **131B**, 121 (1983); J. A. Mignaco and M. A. Rego Monteiro, Phys. Lett. B **175**, 77 (1986).