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Waiting Times for Random Walks on Regular and Fractal Lattices

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We use the exact renormalization procedure, introduced by Machta [Phys. Rev. B **24**, 5260 (1981)], to evaluate the waiting time density for a walker jump to one of its nearest neighbors on a decimated lattice, in terms of the original waiting time densities on the nondecimated lattice. We explicitly identify all the fixed points and their corresponding domain of attraction for the one-dimensional case and for the Sierpinsky gasket.

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A few years ago, Machta[1] introduced a nice, simple renormalization procedure to study the properties of nearest-neighbors random walks in one dimension. Machta was concerned with the properties of random walks with static disorder. More recently, the same procedure was also applied to investigate walks on a one-dimensional quasiperiodic lattice.[2] The idea is to decimate every other site on the 1D lattice, and to calculate the waiting time density (or first-passage time density) to go from a site to its "new" nearest neighbors. These are the next-nearest neighbors on the original lattice. By repeating n such decimations, one obtains first-passage time densities from a given site to its nearest neighbors which are, in fact, at a distance $\pm 2^n$ on the original lattice (lattice constant is 1). To obtain detailed information on the form of this waiting time density as $n \rightarrow \infty$, one has to rescale time in a proper way. It will take a certain factor $\lambda > 1$ more time to get twice as far, and we will therefore scale time down by this factor, at each stage of the decimation. In this way, the waiting time density approaches a scaling form, which we can identify explicitly.

In this Letter, we will use this renormalization procedure for a symmetric nearest-neighbor random walk with a waiting time density $\psi(t)$ which is constant throughout the lattice (and hence so at every stage of the decimation). We will investigate both the 1D lattice and the Sierpinsky gasket. We will identify explicitly the scaling value λ, the fixed points of the renormalization equation, and their corresponding domain of attraction. This has to be contrasted with similar work on the Green's function, whose scaling form is still subject to discussion.[3]

Consider first the 1D lattice. Let $\psi_{(n)}(t)$ be the probability density that a walker, arriving at a site at $t=0$, will move at time t to any one of its nearest neighbors on the n-times decimated lattice. $\psi_{(n)}(t)$ is also the first-passage time density to go from a site to any one of the sites at a distance $\pm 2^n$ on the original lattice. Its Laplace transform reads

$$\tilde{\psi}_{(n)}(s) = \int_0^\infty e^{-st} \psi_{(n)}(t)\, dt . \tag{1}$$

To derive the renormalization equation, we consider $\psi_{(0)}(t)$, the waiting time density on the original lattice, and calculate the waiting time density $\psi_{(1)}(t)$ to one of the next-nearest neighbors. A given next-nearest neighbor can be reached for the first time, and this without passing at the other next-nearest neighbor, in a walk of $2N$ steps, $N=1,2,\ldots$. The $N-1$ first pairs of steps may be taken, starting from the original site, either to the right and back or to the left and back. There are thus 2^{N-1} different walks of length $2N$. The probability for such a walk to take a time exactly equal to t is a convolution of $2N$ factors $\psi_{(0)}(\tau_i)/2$, with $\Sigma \tau_i = t$. In Laplace transform terms, this convolution becomes a product $[\tilde{\psi}_{(0)}(s)/2]^{2N}$. We find thus that $\tilde{\psi}_{(1)}(s)/2$ is the sum of $[\tilde{\psi}^2_{(0)}(s)/2]^N/2$ for N going from one to infinity (the factor $\frac{1}{2}$ comes from the fact that we are looking at the first-passage time to a given neighbor). This result is valid at any stage of the decimation so that we conclude [see also Eqs. (3.3) and (3.4) in Ref. 1 with $\tilde{\psi}=2p=2q$]

$$\tilde{\psi}_{(n)}(s)=\tilde{\psi}^2_{(n-1)}(s)/[2-\tilde{\psi}^2_{(n-1)}(s)]\,. \tag{2}$$

As was explained above, we are now going, after each decimation, to scale time down by a factor λ. The precise value of λ will be determined later. Instead of considering $\psi_{(n)}(t)$, we are interested in $\psi_{(n)}(t/\lambda)/\lambda$. Under Laplace transformation, this corresponds to considering $\tilde{\psi}_{(n)}(s\lambda)$ instead of $\tilde{\psi}_{(n)}(s)$. The renormalization equation, including time rescaling, thus reads

$$\tilde{\psi}_{(n)}(s)=\tilde{\psi}^2_{(n-1)}(s/\lambda)/[2-\tilde{\psi}^2_{(n-1)}(s/\lambda)]\,. \tag{3}$$

The desire is that the successive waiting time densities $\tilde{\psi}_{(n)}(s)$ converge, after a sufficient number of decimations, to a limiting form $\tilde{\psi}(s)$, which is then obviously a fixed point of the transformation (3). Let us first identify these fixed points. The fixed-point equation reads

$$\tilde{\psi}(s)=\tilde{\psi}^2(s/\lambda)/[2-\tilde{\psi}^2(s/\lambda)]\,. \tag{4}$$

This relation can be very much simplified by setting

$$\tilde{\psi}(s)=1/\cosh[\tilde{\phi}(s)]\,. \tag{5}$$

One way to guess this transformation is to remember that the Laplace transformed first-passage time density $\tilde{F}(s)$ to go from 0 to $\pm L$ by diffusion (Brownian motion in 1D), with diffusion coefficient D, is given by

$$\tilde{F}(s)=1/\cosh(L^2s/D)^{1/2}. \tag{6}$$

One can also arrive at the transformation (5) in a more systematic way, as will be illustrated below for the Sierpinsky gasket. By introducing (5) into Eq. (4) we find that

$$\tilde{\phi}(s)=2\tilde{\phi}(s/\lambda)\,. \tag{7}$$

Here, we have disregarded the possibility of an additional multiple of $\pm 2\pi i$ and a different sign in the right-hand side of Eq. (7), since these differences do not change the final result for $\tilde{\psi}=\cosh(\tilde{\phi})$. The general solution of the above equation is given by

$$\tilde{\phi}(s)=A(s)s^{\ln 2/\ln\lambda}. \tag{8}$$

The second factor in the right-hand side of (8) is a particular solution of (7), while

$$A(s)=A(s/\lambda)=\sum_n A_n\exp(2\pi i n\ln s/\ln\lambda) \tag{9}$$

is a function, periodic in $\ln s$ with period $\ln\lambda$, and can hence be represented by its Fourier expansion.[4] This result, together with Eq. (5), specifies all the fixed points. Let us now investigate their domain of attraction.

To do so, we again introduce the transformation (5)

$$\tilde{\psi}_{(n)}(s)=1/\cosh[\tilde{\phi}_{(n)}(s)] \tag{10}$$

and find that the renormalization equation (3) reduces to

$$\tilde{\phi}_{(n)}(s)=2\tilde{\phi}_{(n-1)}(s/\lambda)\,. \tag{11}$$

By iteration, one concludes that

$$\tilde{\phi}_{(n)}(s)=2^n\tilde{\phi}_{(0)}(s/\lambda^n)\,. \tag{12}$$

In the following, we will take $\lambda>1$ (the case $\lambda<1$ leads to a trivial attractor). Hence, it is clear that the renormalized density is, in the limit $n\to\infty$ and for every finite value of s, determined by the behavior of the initial density for $s\to 0$. A large class of waiting time densities are characterized by the following behavior for $s\to 0$ (Ref. 5):

$$\tilde{\psi}_{(0)}(s)\underset{s\to 0}{=}1-\tfrac{1}{2}A_0^2s^\alpha\,, \tag{13}$$

where A_0 and $0<\alpha\le 1$ are constants. These constants are the only information on the initial waiting time density $\psi_{(0)}(t)$ that shows up in the asymptotic limit $n\to\infty$. A_0 fixes the time scale of the walk on the nondecimated lattice. α is the fractal dimension in time[4] of the original waiting time density $\psi_{(0)}(t)$. In particular, for a waiting time density $\psi_{(0)}(t)$ with fractal dimension equal to 1, i.e., finite first moment T, one has

$$\tilde{\psi}_{(0)}(s)\underset{s\to 0}{=}1-Ts\,, \tag{14}$$

corresponding to $\alpha=1$ and $A_0^2=2T$. From Eqs. (10) and (13), it follows that the corresponding result for $\tilde{\phi}_0(s)$ is

$$\tilde{\phi}_{(0)}(s)\underset{s\to 0}{=}A_0s^{\alpha/2}. \tag{15}$$

After a large number of iterations, one converges, according to Eqs. (12) and (15), to the following fixed point:

$$\tilde{\phi}(s)=\lim_{n\to\infty}\tilde{\phi}_{(n)}(s)=\lim_{n\to\infty}A_0(2/\lambda^{\alpha/2})^ns^{\alpha/2}. \tag{16}$$

The scaling that leads to a nontrivial attractor is clearly (see also Ref. 1)

$$\lambda^\alpha=4\,. \tag{17}$$

For $\alpha=1$, the rescaling is by a factor 4, which is precisely what one finds for a usual diffusion process: To go twice as far, it takes 4 times as long. The resulting waiting time densities are then, from Eqs. (5), (16), and (17), explicitly given by

$$\tilde{\psi}(s)=1/\cosh(A_0s^{\alpha/2})\,. \tag{18}$$

Since, according to Eq. (17), $\alpha/2=\ln 2/\ln\lambda$, these correspond to the fixed points, identified in Eqs. (8) and (9), that do not "oscillate wildly" in the limit $s\to 0$, i.e., the only nonvanishing coefficient in Eq. (9) is A_0. We also note that for a scaling not satisfying Eq. (17), one converges to a trivial attractor, namely

$$\lim_{n\to\infty} \tilde{\psi}(s) \equiv \begin{cases} 0, & s\neq 0, \\ 1, & s=0, \end{cases} \quad \lambda^a < 4, \tag{19}$$

and

$$\lim_{n\to\infty} \tilde{\psi}(s) \equiv 1, \quad \lambda^a > 4. \tag{20}$$

The implementation of the above renormalization procedure to regular lattices in higher dimensions seems to be difficult. One reason for this is that the nearest neighbors do not isolate a given point from the rest of the lattice, so that it is very difficult just to derive the renormalization equation, let alone to solve it. But the procedure can be applied to a fractal lattice, more precisely, to the Sierpinsky gasket, as we now proceed to show. The reason for the interest in this example is that its simple structure allows for many detailed calculations (see Refs. 3 and 6 for recent reviews), while it is desired that its fractal nature sheds some light on the behavior of more complicated fractals, such as the percolating cluster. The decimation procedure is the same as that used by previous authors to calculate the spectrum[7–9] or the Green's function.[10,11] For more details, see Fig. 1. The derivation of the renormalization equation, analogous to Eq. (3), is more complicated than for the 1D case, basically because a walker can make very complicated excursions to its two neighboring triangles (0′1′2′ and 0″3′4′ in Fig. 1). We just quote the final result, which turns out to be simple [compare with Eq. (3)]:

$$\tilde{\psi}_{(n)}(s) = \tilde{\psi}^2_{(n-1)}(s/\lambda)/[4-3\tilde{\psi}_{(n-1)}(s/\lambda)]. \tag{21}$$

As in the 1D case, we have introduced a time-rescaling factor λ. The corresponding fixed-point equation reads

$$\tilde{\psi}(s) = \tilde{\psi}^2(s/\lambda)/[4-3\tilde{\psi}(s/\lambda)]. \tag{22}$$

To solve these equations, one can make use of our experience with the 1D case. We set

$$\tilde{\psi}(s) = 1/f[\phi(s)] \tag{23}$$

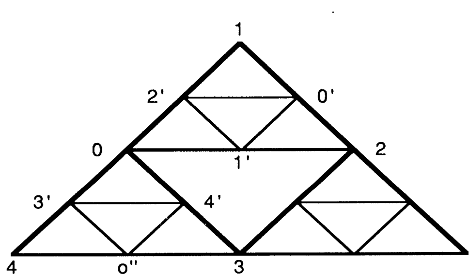

FIG. 1. Decimation on the Sierpinsky gasket. At each step of the decimation, the smaller inner triangles (0′1′2′ and 0″3′4′) are removed.

so that Eq. (22) becomes

$$f[\phi(s)] = 4f^2[\phi(s/\lambda)] - 3f[\phi(s/\lambda)]. \tag{24}$$

In order to simplify this relation further, we look for an analytic function f such that

$$4f^2(x) - 3f(x) = f(ax), \quad \forall x. \tag{25}$$

If such a function can be found (we will derive a Taylor series for f, which also serves as a definition for the function of a complex argument), we obtain from Eq. (24) that

$$\phi(s) = a\phi(s/\lambda). \tag{26}$$

This equation can be solved in the same way as Eq. (7). Restricting ourselves to functions that are "well behaved" at $s=0$, one finds from Eq. (26)

$$\phi(s) = \tfrac{1}{2} A_0^2 s^{\ln a/\ln\lambda}, \tag{27}$$

where A_0 is an arbitrary constant [the prefactor in Eq. (27) has been chosen such as to be consistent with the definition of A_0 through Eq. (13), see below]. The further discussion proceeds along the same lines as for the 1D case.

Before proceeding with the results, we show that the function f exists and is, in a certain sense, unique. From Eq. (25), it follows that, if f exists and is not equal to the trivial solution $f(x)=0$, then $f(0)=1$. If it has a first derivative $f'(0)\neq 0$ at $x=0$, it follows that $a=5$. We anticipate now, on what is following, by noting that all the higher derivatives, $f^{(n)}(0)$ at $x=0$, are proportional to $[f'(0)]^n$. The case $f'(0)=0$ leads then to the trivial solution $f\equiv 1$. On the other hand, for $f'(0)\neq 0$, we can, without loss of generality, assume that $f'(0)=1$, since a factor different from 1 can always be incorporated into the definition of ϕ; cf. Eq. (23). The calculation of the higher-order derivatives $f^{(n)}(0)$ goes as follows. From $f(0)=f'(0)=1$ and Eq. (25) with $a=5$, one finds that, for $n\geq 2$

$$(5^n-5)f^{(n)}(0) = 8\sum_{r=1}^{n-1} \binom{n-1}{r} f^{(r)}(0) f^{(n-r)}(0). \tag{28}$$

The derivative $f^{(n)}$ can thus be calculated in terms of the lower-order derivatives. For example, one has

$$f(0)=f'(0)=1, \quad f''(0)=\tfrac{2}{5}, \\ f'''(0)=\tfrac{2}{25}, \quad f''''=\tfrac{8}{775}, \quad \dots \tag{29}$$

We have thus constructed explicitly a solution to Eq. (25) and showed that it is essentially unique [apart from the trivial solution, and the choice of $f'(0)=1$]. From Eq. (27), it furthermore follows by induction that $0<f^{(n)}(0)\leq 1$, $\forall n$, and hence the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{30}$$

converges for all x. This series also defines the corresponding function of a complex argument. This completes our proof of the properties of f which are needed to establish the relation Eq. (26).

We now turn back to state the limit theorems for the waiting time derivatives on a decimated Sierpinsky gasket. The relevant time-rescaling factor λ is given by

$$\lambda^{\alpha}=5\,. \tag{31}$$

With the rescaling, a waiting time distribution $\tilde{\psi}_{(0)}(s)$ of type (13) will converge, upon renormalization on a Sierpinsky gasket, cf. Eq. (21), to the limiting fixed-point density:

$$\tilde{\psi}(s)=1/f(\tfrac{1}{2}A_0^2 s^{\alpha})\,, \tag{32}$$

where f is the function defined above. The differences with the 1D scaling are thus the different rescaling given by Eq. (31) rather than Eq. (17), and the more complicated form of the final attractors [the 1D case corresponds to $f(x)=\cosh(2x)^{1/2}$]. For a waiting time distribution with a finite first moment, such as for Markovian walks, one has $\alpha=1$. Hence one has to "wait" five times longer to "go" twice as far. This is, of course, due to the fractal nature of the lattice. It is in agreement with the following result for the radial displacement on the Sierpinsky gasket[12–15]:

$$(\langle r^2(t)\rangle)^{1/2}\sim t^{\ln 2/\ln 5}\,. \tag{33}$$

Since Eq. (32) is the central result of our paper, we reformulate it in a more explicit way for a random walk with finite first moment $T=1$ (this choice fixes the time scale). The Laplace transform of the first-passage time density to go from a site on the Sierpinsky gasket to one of its four neighbors at a distance of 2^n, $\tilde{D}_{\rm FPT}(0\rightarrow 2^n,s)$, attains, for n large, the following scaling form:

$$\lim_{n\rightarrow\infty}\tilde{D}_{\rm FPT}(0\rightarrow 2^n,s/5^n)=1/f(s)\,. \tag{34}$$

The function f is the unique solution of Eq. (25) with $a=5$, $f(0)=f'(0)=1$. More explicitly, f is given by the series expansion Eq. (30), whose coefficients can be obtained recursively from Eq. (28). The knowledge of the small-s expansion of f is sufficient to generate the moments $\langle\tau^n\rangle$ of the first-passage time:

$$\langle\tau^n\rangle=\int_0^{\infty}\tau^n\psi(\tau)d\tau=(-1)^n\frac{d^n\tilde{\psi}(s)}{ds^n}\bigg|_{s=0}\,. \tag{35}$$

For example, we obtain

$$\langle\tau^2\rangle=\tfrac{8}{5}\langle\tau\rangle^2,\quad\langle\tau^3\rangle=\tfrac{92}{25}\langle\tau\rangle^3\,,\qquad \langle\tau^4\rangle=\tfrac{8672}{775}\langle\tau\rangle^4\,. \tag{36}$$

This leads to a value $\langle\delta\tau^2\rangle/\langle\tau\rangle^2=\frac{3}{5}$, to be compared with the value of this ratio equal to $\frac{2}{3}$ for the 1D case. Further information on the properties of f can be derived from the observation that it allows to express the solution of the discrete mapping $x_n=4x_{n-1}^2-3x_{n-1}$ as $x_n=f(5^n f^{-1}(x_0))$.

Let us close with a final remark. The limit theorems that we have proven here deal with first-passage times densities. On the other hand, there exists extensive literature on the convergence of probability densities, i.e., the convergence of random walks to diffusion processes (central limit theorems). Obviously, there must exist a link between these two, which we have not further explored in this Letter. If this relationship is not too complicated, the results obtained in this Letter may be used to settle the question of the scaling form of the Green's function on the Sierpinsky gasket. Finally, we mention that the approach presented here can also be applied to other simply connected deterministic fractals.

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