Dynamical Superalgebra of the "Dressed" Jaynes-Cummings Model

C. Buzano,[(1)] M. G. Rasetti,[(1)] and M. L. Rastello[(2)]

[(1)]*Dipartimento di Fisica and Unità Consorzio Interuniversitario di Struttura della Materia, Politecnico, Torino, Italy*

[(2)]*Instituto Elettrotecnico Nazionale "G.Ferraris," Torino, Italy*

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We show how, by resorting to a one-fermion realization of the Pauli operators, the Hamiltonian of the Jaynes-Cummings (JC) model can be identified as an element of the superalgebra u(1 | 1), which plays the role of a dynamical algebra. The extension of this notion to osp(2 | 2) allows adding both virtual and real two-photon processes to the JC Hamiltonian. The exact diagonalization problem is tackled here in the special case when the coupling constants of the fermionic terms of the "dressed" JC Hamiltonian are assumed to nilpotent Grassman-Banach numbers.

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The idealization of the fundamental two-level atom and single-mode radiation field interaction provided by the Jaynes-Cummings (JC) model[1] matches, with unexpected accuracy, Rydberg maser experiments designed to detect the atom-single-photon coupling.[2] However, it requires consistent modifications in order to encompass the features of a ground test on quantum electrodynamics. The fundamental minimal coupling form of the Hamiltonian for the interaction of light with a bound electron is

$$H=\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2+U(\mathbf{r})+\frac{1}{8\pi}\int d^3\mathbf{x}(\mathbf{E}^2+\mathbf{B}^2)$$
$$=H_{\rm el}+H_{\rm field}+H_{\rm int}\,, \tag{1}$$

where $H_{\rm el}$ is the energy of the electron in the absence of the electromagnetic field [the potential energy being $U(\mathbf{r})$], whereas

$$H_{\rm int}=(e/2mc)(-\mathbf{A}\cdot\mathbf{p}-\mathbf{p}\cdot\mathbf{A}+\mathbf{A}^2/c)\,, \tag{2}$$

and the Coulomb gauge is assumed for the vector potential $\mathbf{A}$.

In the dipole approximation—justified by the relative magnitude of the atom size and of the optical wavelength[3]—the Hamiltonian (1) of an atom interacting with a single-mode radiation field can be written, under the assumption of a two-level atom, in a second quantized scheme, as the sum of

$$H_{\rm el}=\tfrac{1}{2}\,\omega(a_2^\dagger a_2-a_1^\dagger a_1)\,, \tag{3a}$$

$$H_{\rm field}=\nu(b^\dagger b+\tfrac{1}{2})\,, \tag{3b}$$

$H_{\rm int}=H_{\rm int}^{(1)}+H_{\rm int}^{(2)}$, where

$$H_{\rm int}^{(1)}=gba_2^\dagger a_1+gb^\dagger a_2^\dagger a_1+\text{H.c.}\,, \tag{3c}$$

$$H_{\rm int}^{(2)}=\kappa(b^\dagger+b)^2\,. \tag{3d}$$

In Eqs. (3a)-(3d) $b^\dagger$ and b denote the photon creation and annihilation operators satisfying the commutation relation $[b,b^\dagger]=I$; $a_i^\dagger$ and a_i, $i=1,2$, denote the creation and annihilation operators for the electron in the state corresponding to level i satisfying the anticommutation relations $\{a_i,a_j\}=0$, $\{a_i,a_j^\dagger\}=\delta_{ij}I$; $\hbar$ was set equal to 1; and the frequencies ω and ν, as well as the coupling constant κ, were assumed real. Naturally the operators $b,b^\dagger$ commute with $a_i,a_i^\dagger$, $i=1,2$.

$H_{\rm int}^{(1)}$ consists of two terms: The first term represents real transitions (where, say, the electron jumps from the lower level 1 to the upper level 2 as a photon is absorbed, or the conjugate process), and the second describes virtual transitions referring to processes in which one photon is created and then absorbed while the electron goes from level 1 to level 2 and back (or vice versa). Such virtual transitions should be considered as a single two-photon process.

In a rotating-wave approximation the second term of $H_{\rm int}^{(1)}$ is related to high-frequency phenomena, compared with the slowly varying processes typical of the direct single-photon transitions. Thus its average contribution over macroscopic times is negligible. One should, however, keep in mind that just virtual processes of this kind are responsible for the energy-level shift in hydrogenlike atoms.[4]

As for $H_{\rm int}^{(2)}$, since it stems out of $\mathbf{A}^2$, it describes two-photon processes whose contribution is of the same order of magnitude as the above virtual processes.[5]

The customary JC model derives from (1) and (3), when the rotating-field approximation is adopted, and all the two-photon processes are neglected. It is usually written, resorting to the two-fermion realization of su(2), $\sigma_+=(\sigma_-)^\dagger=a_2^\dagger a_1$, $\sigma_z=\frac{1}{2}(a_2^\dagger a_2-a_1^\dagger a_1)$, in the form

$$H_{\rm JC}=\omega\sigma_z+\nu(b^\dagger b+\tfrac{1}{2})+gb^\dagger\sigma_-+\bar{g}b\sigma_+\,, \tag{4}$$

where $\sigma_\pm=\sigma_x\pm i\sigma_y$ and σ_α, $\alpha=x,y,z$ denotes the Cartesian components of a spin-$\frac{1}{2}$ angular momentum operator (equal to $\frac{1}{2}$ of the corresponding Pauli matrices).

In this Letter we describe an approach to the JC model, resorting to the concept of dynamical superalgebra,[6] to show that not only can different regimes of the model be described within a unified scheme, but also the higher-order terms— neglected in order to derive it— can be reinserted and taken into account by the same (super)algebraic structure.

The notion of dynamical superalgebra is the following. If the Hamiltonian H is recognized as an element of some n-dimensional rank-r Lie algebra $\mathcal{L}$, the spectrum is obtained in a straightforward way by means of an automorphism Φ: $\mathcal{L} \to \mathcal{L}$ such that $\Phi(H) = \sum_{j=1}^{r} \alpha_j h_j$, where the set $\{h_1, \ldots, h_r; e_1, \ldots, e_{n-r}\}$ is a Cartan basis for $\mathcal{L}$. $\mathcal{L}$ is said to be the dynamical (or spectrum generating) algebra for H if Φ is an inner automorphism. A Lie superalgebra $\mathcal{S}$, on the other hand, is but a $\mathbb{Z}_2$-graded Lie algebra, namely one in which the generators are classified as even (bosonic) and odd (fermionic) with the property that the product of two operators with the same parity is even, whereas the product of an odd operator by an even operator is odd. The bosonic generators define a Lie algebra $\mathcal{S}_0$ by commutation, the fermionic generators are tensor operators corresponding to some representation of $\mathcal{S}_0$ and satisfy anticommutation relations.

The basic step of our approach is the observation that by the Holstein-Primakoff method,[7] the realization of su(2) in terms of two fermions $a_i, a_i^\dagger$, $i = 1,2$ (or, equivalently, of Pauli matrices) can be reduced to one in terms of a single fermionic mode $(f, f^\dagger)$ by setting $\sigma_+ = f^\dagger$, $\sigma_- = f$, and $\sigma_z = \frac{1}{2}(2f^\dagger f - I)$, where $\{f,f\} = 0$, $\{f, f^\dagger\} = I$. Furthermore, after the change in notation $b \equiv B_-, b^\dagger \equiv B_+, f \equiv F_-, f^\dagger \equiv F_+$, using the customary mathematical notation concerning superalgebras, we notice that the superalgebra sh(1) generated by $\{F_+, F_-; B_+, B_-, I\}$, with the commutation-anticommutation relations $\{F_\epsilon, F_\eta\} = \delta_{\epsilon,-\eta} I$, $[B_-, B_+] = I$, $[F_\epsilon, B_\eta] = 0$, $(\epsilon, \eta = \pm)$ (often referred to as the super-Heisenberg algebra) is but the graded version of the Weyl-Heisenberg algebra. Then, upon defining

$$V = F_+F_- + B_+B_-, \quad M = F_+F_- - B_+B_- - I, \tag{5a}$$

$$Q_+ = Q_-^\dagger = 2^{-1/2} B_- F_+, \tag{5b}$$

the JC Hamiltonian is

$$H_{\rm JC} = \tfrac{1}{2}(\omega + \nu)V + \tfrac{1}{2}(\omega - \nu)M + \Gamma Q_+ + Q_- \bar{\Gamma} \tag{6}$$

and can be easily recognized as an element of the superalgebra associated with the unitary supergroup u(1|1). Indeed the tensor operators (5) in the enveloping algebra of sh(1) satisfy the commutation relations $\{Q_\epsilon, Q_\eta\} = \frac{1}{2} V \delta_{\epsilon,-\eta}$, $[V,M] = 0 = [V, Q_\epsilon]$, $[M, Q_\epsilon] = 2\epsilon Q_\epsilon$ from which it appears that the bosonic sector algebra $\mathcal{S}_0$ is u(1)⊕u(1), whereas M acting on the fermionic subspace generates another so(2) ≈ u(1) subalgebra. V is the Casimir operator.

Two observations are now in order. $H_{\rm JC}$ is indeed a bosonic element, as it should be, because the overlap integral g, when we switched from (4) to (6) has itself acquired an odd gradation (it is related now to the quantum amplitude of a fermionic field). We denoted it as Γ in order to emphasize this new feature. Γ should be treated not as a c number, but as an anticommuting variable (anticommuting also with fermion operators), belonging, e.g., to the odd part $\mathbb{Q}_1$ of the $\mathbb{Z}_2$-graded Banach-Grassman algebra $\mathbb{Q} = \mathbb{Q}_0 \oplus \mathbb{Q}_1$.[8] Notice that $\Gamma\bar{\Gamma}$ can be considered as a c number, in that it commutes with everything else. In order to illustrate more clearly the algebraic method, we shall henceforth treat it as nilpotent, even though this is not strictly necessary, and somewhat conceals its physical meanings. Indeed $\Gamma\bar{\Gamma}$ could be assumed to be a complex variable. In order to derive a numerical value for it one should, e.g., consider the JC atom as immersed in a heat bath at fixed temperature T, and solve the corresponding self-consistency equations.[6] On purely quantum mechanical grounds, for an isolated atom, $\Gamma\bar{\Gamma}$ should be regarded simply as a spectral parameter.

The second observation is that the group space in which the rotation $\mathcal{U}$ corresponding to the automorphism Φ is realized, in this case, is in fact a superspace. $\mathcal{U}$ maps the fermionic sector onto the bosonic one, and is implemented by the adjoint action $\exp(adZ)$ of the skew-Hermitian operator $Z = (\Psi Q_+ + \bar{\Psi} Q_-)$, whose characteristic "angle" $\Psi \in \mathbb{Q}_1$. Upon selecting $\Psi = (\omega - \nu)^{-1}\Gamma$, one gets for $\nu \neq \omega$,

$$\begin{aligned} H_d &= \exp(adZ)(H_{\rm JC}) \\ &\equiv \sum_{n=0}^{\infty} \frac{1}{n!}[Z,[Z,\ldots[Z,H_{\rm JC}]]\ldots] \ (n \text{ brackets}) \\ &= \{\tfrac{1}{2}(\omega+\nu) + [2(\omega-\nu)]^{-1}\Gamma\bar{\Gamma}\}V + \tfrac{1}{2}(\omega-\nu)M\,. \end{aligned} \tag{7}$$

Notice that (7) is precisely the result one obtains from the customary JC spectrum,[1] by setting $g\bar{g} = \pm\frac{1}{2}\Gamma\bar{\Gamma}$ and expanding the square root as if the latter were nilpotent.

If $\nu = \omega$, $H_{\rm JC}$ is but an element of the central extension of the Weyl-Heisenberg algebra generated by $A = \Gamma Q_+$, $A^\dagger = Q_-\bar{\Gamma}$, $C = \Gamma\bar{\Gamma}V$ (with commutation relations $[A, A^\dagger] = C$, $[A,C] = 0 = [A^\dagger, C]$), and diagonalization— achieved in straightforward way— gives

$$H_d = \omega V + \xi V^{1/2}\Gamma\bar{\Gamma}, \tag{8}$$

where ξ is the scale-fixing constant.

We now observe that there is a complete chain of finite-dimensional simple Lie superalgebras embedded one into the other, each generated by tensor operators realized in the enveloping algebra of sh(1), rooted in u(1|1) (Ref. 9):

$$\mathrm{u}(1|1) \subset \mathrm{osp}(1|2) \subset \mathrm{osp}(2|2) \subset \ldots,$$

whose dimensions are respectively 4,5,8, ... osp(2|2) is generated by $\{Q_+,Q_-,C_+,C_-;K_+,K_-,K_0,F_0\}$, where the new fermionic elements are given by

$$C_\epsilon = 2^{-1/2} B_\epsilon F_\epsilon = C^\dagger_{-\epsilon}, \quad \epsilon = \pm\,, \tag{9}$$

the new bosonic elements by

$$K_\epsilon = (2\sqrt{2})^{-1} B_\epsilon^2 = K^\dagger_{-\epsilon}, \quad \epsilon = \pm\,, \tag{10}$$

and the linear combinations

$$K_0 = \tfrac{1}{2}(B_+B_- + \tfrac{1}{2}I) = \tfrac{1}{4}(V-M)\,, \tag{11a}$$

$$F_0 = \tfrac{1}{2}(F_+F_- - \tfrac{1}{2}I) = \tfrac{1}{4}(V+M)\,, \tag{11b}$$

were introduced for convenience.

The bosonic commutation relations are $[K_-,K_+] = K_0$, $[K_0,K_\epsilon] = \epsilon K_\epsilon$, $[F_0,K_\epsilon] = 0$, and $[F_0,K_0] = 0$. The fermionic anticommutation relations are $\{Q_\epsilon,Q_\eta\} = \delta_{\epsilon,-\eta}(K_0+F_0)$, $\{C_\epsilon,C_\eta\} = \delta_{\epsilon,-\eta}(K_0-F_0)$, and $\{Q_\epsilon,C_\eta\} = \sqrt{2}\delta_{\epsilon,-\eta}K_\eta$. Moreover, we have the mixed boson-fermion commutators $[Q_\epsilon,K_\eta] = 2^{-1/2}\epsilon\delta_{\eta,\epsilon}C_\epsilon$, $[C_\epsilon,K_\eta] = 2^{-1/2}\eta\delta_{\epsilon,-\eta}Q_\epsilon$, $[Q_\epsilon,K_0] = \tfrac{1}{2}\epsilon Q_\epsilon$, $[C_\epsilon,K_0] = -\tfrac{1}{2}\epsilon C_\epsilon$, $[F_0,C_\epsilon] = \tfrac{1}{2}\epsilon C_\epsilon$, and $[F_0,Q_\epsilon] = \tfrac{1}{2}\epsilon Q_\epsilon$. One can easily check that $\{K_+,K_-,K_0\}$ generate a $\mathrm{sp}(2)\sim\mathrm{su}(1,1)$ subalgebra, whereas F_0 acting on the fermionic subspace span $\{Q_+,Q_-,C_+,C_-\}$ generates a further $\mathrm{so}(2)\sim\mathrm{u}(1)$ subalgebra. The realizations (5b), (9), (10), and (11) correspond to the Casimir operator

$$\mathscr{C} \equiv K_0^2 - \{K_+,K_-\} - F_0^2 - F_0 + Q_+Q_- - C_+C_-$$

being equal to zero and the operators Q_ϵ, C_ϵ anticommuting with F_0: $\{Q_\epsilon,F_0\} = 0 = \{C_\epsilon,F_0\}$. The most general Hamiltonian which is a bosonic element of $\mathrm{osp}(2\,|\,2)$ [more precisely $\mathrm{osp}(2\,|\,2)+\mathrm{sh}(1)$] can now be written as

$$H = H_{\mathrm{JC}} + (\gamma K_+ + \bar{\gamma}K_-) + (\Lambda C_+ + C_-\bar{\Lambda})\,, \tag{12}$$

where $\gamma \in \mathbb{C}$ and $\Lambda \in \mathbb{Q}_1$ (notice that in the new notation $H_{\mathrm{JC}} = 2\omega F_0 + 2\nu K_0 + \Gamma Q_+ + Q_-\bar{\Gamma}$, and that also $\Lambda\bar{\Lambda}$ will be assumed to be nilpotent).

It is easy to recognize that the two new terms in (12) are just those previously neglected, respectively, $H^{(2)}_{\mathrm{int}}$ (provided $\gamma = 2\sqrt{2}\kappa$ and the frequency ν is increased by 2κ) and the second factor in $H^{(1)}_{\mathrm{int}}$ (with $\Lambda = \Gamma$). Thus the whole Hamiltonian (1) is described in second-quantized form and in the single-fermion realization discussed above by (12) (referred to in Ref. 10 as the "dressed" JC model). The adjoint action of the skew-Hermitian operator $Z = \Phi Q_+ + \bar{\Phi}Q_- + \Theta C_+ + \bar{\Theta}C_-$, where Φ, $\Theta \in \mathbb{Q}_1$, by selecting

$$\Phi = \frac{\omega+\nu}{\delta}\Gamma - \frac{1}{\sqrt{2}}\frac{\bar{\gamma}}{\delta}\Lambda + \frac{\sqrt{2}}{3}\frac{\bar{\gamma}}{\delta^2}\Lambda\Gamma\bar{\Gamma} + \frac{2}{3}\frac{2\omega+\nu}{\delta^2}\Gamma\Lambda\bar{\Lambda}\,,$$
$$\tag{13}$$
$$\Theta = \frac{\omega-\nu}{\delta}\Lambda + \frac{1}{\sqrt{2}}\frac{\gamma}{\delta}\Gamma + \frac{\sqrt{2}}{3}\frac{\gamma}{\delta^2}\Gamma\Lambda\bar{\Lambda} - \frac{2}{3}\frac{2\omega-\nu}{\delta^2}\Lambda\Gamma\bar{\Gamma}\,,$$

where $\delta = \omega^2 - \nu^2 + \tfrac{1}{2}|\gamma|^2$, rotates H into a new Hamiltonian $H_B \in \mathscr{B}(\mathrm{osp}(2\,|\,2))$,

$$H_B = 2\omega_B F_0 + 2\nu_B K_0 + \sigma K_+ + \bar{\sigma}K_-\,. \tag{14}$$

Upon setting $\alpha_\pm = [(\omega+\nu)\Gamma\bar{\Gamma} \pm (\omega-\nu)\Lambda\bar{\Lambda}]/2\delta$, $\beta = (1/\delta^2)\Gamma\bar{\Gamma}\Lambda\bar{\Lambda}$, and $\rho = (\gamma/\sqrt{2}\delta)\Gamma\bar{\Lambda}$, the coefficients are given by $\omega_B = \omega(1+\beta) + \alpha_- - \mathrm{Re}(\rho)$, $\nu_B = \nu(1+\tfrac{1}{2}\beta) + \alpha_+$, and

$$\sigma = \gamma[1 + \tfrac{1}{2}\beta + (\omega\alpha_- - \nu\alpha_+)/(\omega^2-\nu^2) + 2\omega\rho/\gamma^2]\,.$$

Customary rotation in $\mathrm{su}(1,1)$ gives finally

$$H_d = 2\omega_d F_0 + 2\nu_d K_0$$
$$= \omega_d F_+F_- + \nu_d B_+B_- + \tfrac{1}{2}(\nu_d - \omega_d)I\,,$$

with $\omega_d = \omega_B$ and $\nu_d = (\nu_B^2 - |\sigma|^2/2)^{1/2}$. The eigenvalues are straightforwardly obtained in the direct-sum Fock space of bosons and fermions (F_+F_- has eigenvalues 0,1 and B_+B_- has eigenvalues $n = 0,1,2,\ldots$). In the case corresponding to the "dressed" JC model ($\Lambda \equiv \Gamma$), the spectrum is completely determined by

$$\omega_d^{(\mathrm{JC})} = \omega + [(\nu - \mathrm{Re}\gamma/\sqrt{2})/\delta]\Gamma\bar{\Gamma}\,,$$

$$\nu_d^{(\mathrm{JC})} = (\nu^2 - |\gamma|^2/2)^{1/2} + \frac{\omega(\nu - \mathrm{Re}\gamma/\sqrt{2})}{\delta(\nu^2 - |\gamma|^2/2)^{1/2}}\Gamma\bar{\Gamma}\,,$$

thus proving the exact diagonalizability of the quantum version of (1) for a two-level atom, at least for $\Gamma\bar{\Gamma} \in \mathbb{Q}_0$. The case in which $\Gamma\bar{\Gamma}$ and $\Lambda\bar{\Lambda}$ are not assumed to be nilpotent will be discussed elsewhere.

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