Fluctuations and Scaling in a Model for Boundary-Layer-Induced Turbulence

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A simple model, in which a boundary layer creates spatial turbulence by emitting "hot plumes," is introduced. In the turbulent regime we find exponentially decaying fluctuations, a characteristic frequency of time signals, and a power-law scaling between analogs of hydrodynamic quantities.

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Boundary layers can play a crucial role for the onset of turbulence in a fluid.[1] The layers occur because a fluid is subjected to boundary conditions for the velocity and temperature fields at the boundaries of a container. In some convection experiments the fluid exhibits a thermal boundary layer characterized by a strong thermal gradient as compared to the gradient in the body of the fluid. With an increase of the gradient, the boundary layer becomes unstable against wave formations and wave fronts may detach from the boundary layer and be convected as "plumes" into the laminar flow.[2] The motion of these plumes causes large fluctuations in the "hard" turbulent regime.

Generally, it is difficult to calculate this spatiotemporal behavior in the turbulent state from the Navier-Stokes equations. In order to gain intuition and facilitate computations, models with a set of coupled dynamical systems, coupled-map lattices, have recently been introduced.[3-5] Here we shall adopt a similar approach with the addition that a "thermal" boundary layer is introduced into the model. This is achieved by keeping a fixed "hot" boundary condition at the bottom plate of the container. Then a boundary layer will appear as time evolves. At a certain critical value of a gradient across the system the boundary layer becomes unstable and hot plumes are released. The laminar state is then locally excited into turbulent behavior by the plumes which are convected some distance into the laminar state. This distance is determined by the strength of the heat diffusion and the turbulent fluctuations that build up in the center of the container and therefore a consequence of the interplay between heat diffusion and heat convection.

A study of turbulent versus nonturbulent behavior via dynamical systems calls for maps that exhibit both chaotic and regular ("laminar") behavior. Very simple maps of this type have recently been introduced by Chaté and Manneville[6] and are of the form

$$f_r(x) = \begin{cases} rx\,, & x \le 0.5\,, \\ r(1-x)\,, & 0.5 < x \le 1\,, \\ x\,, & x > 1\,. \end{cases} \tag{1}$$

For $r > 2$ the map (1) is a chaotic repeller. The motion is chaotic when $x \le 1$, which we denote "hot." The transient state, $x_L > 1$, is laminar in the sense that $f_r(x_L) = x_L$, and this state is always marginally stable. When maps of the form (1) are placed on a 1D lattice and coupled diffusively, Chaté and Manneville found that as the coupling exceeds a critical value, hot sites percolate through the system and give rise to spatio-temporal intermittency patterns.[6]

Here we introduce a similar model in two dimensions. The crucial new ingredient is the addition of a convection term:

$$x_{n+1}^{(i,j)} = f_r(x_n^{(i,j)}) + \tfrac{1}{4}\epsilon[f_r(x_n^{(i-1,j)}) + f_r(x_n^{(i+1,j)}) + f_r(x_n^{(i,j-1)}) + f_r(x_n^{(i,j+1)}) - 4f_r(x_n^{(i,j)})] + v[f_r(x_n^{(i,j-1)}) - f_r(x_n^{(i,j)})]\,. \tag{2}$$

(i,j) is a point on an $N \times N$ lattice and n is the time step. The parameter ϵ controls the diffusion and the v term models the convection that displaces hot fluid vertically.[7] The boundary layer is enforced by the constraint that the system is kept hot at the bottom $(j=1)$

$$x^{(i,1)} = x_B\,, \quad x_B < 1\,. \tag{3}$$

The boundary conditions at the other three boundaries are free. We set $x_B = 0$ and the parameter in (1) to $r = 3$. In the calculations presented here, the value of ϵ is less than the critical value ϵ_c because hot sites should not percolate but only be induced diffusively from the boundary. The model is initialized in a state where all sites are laminar $x_0^{(i,j)} = 1.1 + \eta$ (η represents a small amplitude noise term). The time evolution is then visualized by marking the hot sites, i.e., sites where $x^{(i,j)} \le 1$. In that way it is easy to observe the hot plumes that travel through the laminar state. If a site is not excited into a hot site by a convecting plume, it stays laminar (i.e., $x^{(i,j)} > 1$). After a few time steps hot sites will diffuse a short distance from the hot plate into the system. This is manifested as a boundary layer where heat is only transported by diffusion and this layer is identified in the following way. For a specific value of ϵ we find a critical value v_c below which there is a sharp gradient (over a few lattice layers) of the number of hot sites in each layer and no gradient in the number of hot sites above these layers, which is the laminar state. This is the analog of a sharp temperature gradient in a thermal boundary layer of a convection experiment. To be more specific: The number of hot sites in layer j is calculated and denoted $H(j)$. Then, if $v < v_c$, $H(j)$ $(<N)$ falls off to zero within a few layers above $j=1$. As the strength of the v term is increased beyond the critical value the layer becomes unstable and plumes (i.e., patches where $x^{(i,j)} < 1$) are constantly released from the boundary layer. The plumes are convected upwards but due to diffusion most of them diminish in size and eventually disappear. Some plumes reach the top of the container. This is illustrated in Fig. 1.

The turbulent state can be characterized quantitatively by its fluctuations. To measure those we place a probe at a specific point in the center of the cell [here at $(i,j)=(25,18)$ with $N=50$; see Fig. 1] and measure the number of time steps t_p for each plume to pass the probe. As the system evolves many plumes sweep intermittently across the probe. The corresponding distribution of times $D(t_p)$ is plotted in Fig. 2 on a semilogarithmic scale. The straight line indicates an exponential distribution

$$D(t_p) \sim \exp(-at_p). \qquad (4)$$

To check for universality the distribution is calculated for four different values of v and ϵ. When normalized as in Fig. 2 there does not seem to be any significant dependence in the constant a on the parameters v and ϵ. Surprisingly, the same constant is found even far above the percolation threshold (at $v=0.08$ shown by the squares in Fig. 2). Similar exponential behavior has also been observed experimentally in the hard turbulent regime.[2] As argued in Ref. 2, one expects an exponential distribution when the behavior around the probe can be considered as a series of randomly distributed events. This is very much in accordance with our visualization of the motion around the probe; plumes pass in an intermittent fashion and one event is little influenced by the previous events.

In the experimental and theoretical work of Heslot, Castaing, and Libchaber[8] and of Castaing *et al.*,[2] the scaling behavior between various hydrodynamic quantities was investigated. To adopt a similar approach for this system we consider a quantity that is a measure of the total heat flux fed into the system. This quantity is estimated by $H(2)$, the total number of hot sites in the layer adjacent to the bottom layer.[9] Averaging over many time steps ($\sim$2500) and dividing by the size of the system, we obtain the fraction of hot sites $\langle H(2)\rangle/N \equiv \langle H\rangle$. The gradient term, proportional to v, plays the role of a temperature gradient over the system. We now vary v and calculate the corresponding fraction of hot sites $\langle H\rangle$. Figure 3 shows the results indicating a power-law scaling

$$\langle H\rangle \sim v^{\beta} \qquad (5)$$

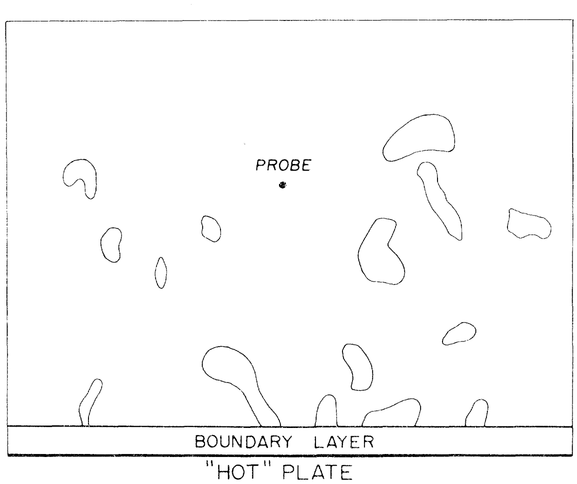

FIG. 1. A snapshot of the simulation calculated for $\epsilon=0.12$, $v=0.04 > v_c \sim 0.018$, $r=3.0$, and $x_B=0$ on a 50×50 lattice. The patches are the "hot plumes" for which $x^{(i,j)} < 1$. The plumes are released from the boundary layer and drift convectively upwards.

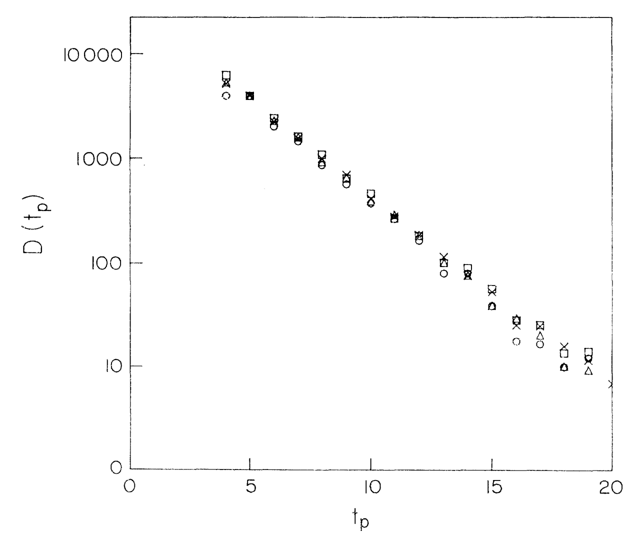

FIG. 2. The distribution of passage times $\log D(t_p)$ vs t_p. Circles: $\epsilon=0.12$, $v=0.035$; triangles: $\epsilon=0.12$, $v=0.05$; squares: $\epsilon=0.12$, $v=0.08$; crosses: $\epsilon=0.14$, $v=0.05$. The different curves are normalized to the same value of $D(t_p)$ at $t_p=5$. Measurements for $t_p \lesssim 3$ are disregarded. Each calculation is performed over $\sim 10^6$ time steps.

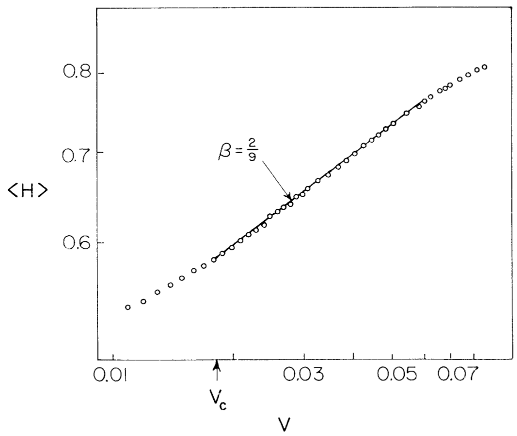

FIG. 3. A plot for $\epsilon=0.12$ of the fraction of hot sites $\langle H\rangle$ vs the gradient v on a log-log scale. Each point of $\langle H\rangle$ is averaged over 2500 time steps. The line has a slope of $\frac{2}{9}$. The critical value v_c, where the boundary layer becomes unstable, is indicated by an arrow.

from the critical value v_c and up to $v \sim 0.06$. The line through the points has a slope $\beta=\frac{2}{9}$ and fits the points quite well (although there is no immediate reason for the exponent to be a rational number). For v below v_c, the boundary layer is stable and plumes are not emitted, which is seen as a deviation from the $\beta=\frac{2}{9}$ scaling law. For $v > \sim 0.06$ the density of the plumes becomes very high and they start to percolate through the laminar regime. This changes the motion of the plumes and the numerical measurements indicate a deviation from the scaling law (see Fig. 3). A percolating density of plumes should be characterized by a very large value of a corresponding Rayleigh number such that the range of scaling in Fig. 3 might model many orders of magnitude in Rayleigh number. It is therefore tempting to draw a parallel between the scaling law Eq. (5) and the power-law scaling between Nusselt and Rayleigh numbers measured in convection (Refs. 2 and 8). However, variation in v changes the effective heat diffusion and $\langle H\rangle$ cannot be considered to vary proportionally to a Nusselt number. Similarly, a change in v is not likely to be proportional to a change in a Rayleigh number. Qualitatively, however, there is similarity between the model and the experiment. The exponent β might change if the model is investigated in 3D instead of in 2D. Also, the exponent is likely nonuniversal.[10]

Finally, the frequency spectrum of a time signal recorded at a probe is considered. At a point (i,j) on the lattice we monitor a signal $y_n^{(i,j)}$ defined as

$$y_n^{(i,j)}=\begin{cases}0 \text{ if } x_n^{(i,j)} \le 1\,, \\ 1 \text{ if } x_n^{(i,j)} > 1\,. \end{cases} \tag{6}$$

The signal is digitized to avoid "noise" from the chaotic motion where $x_n^{(i,j)}$ jumps around in the interval [0,1] as time evolves. Time sequences (~ 10000 steps) have been numerically extracted at two probes, one close to the bottom plate at $(i,j)=(25,3)$, and one at the central probe $(i,j)=(25,18)$. The corresponding Fourier spectrum shows a clear maximum, defining a characteristic frequency of the motion. In agreement with the experimental observations,[2] the characteristic frequency at the center ω_c is lower than the characteristic frequency at the bottom, ω_b. We find evidence for a scaling law, $\omega_b \sim v^{\delta}$, with δ close to 0.5, but our numerical results here are not as good as the scaling law for the heat flux depicted in Fig. 3.

In conclusion, we have introduced a simple model for spatio-temporal turbulence with a boundary layer. At a critical value for the heat flux, the boundary layer becomes unstable and emits patches of the layer into the laminar regime. The associated fluctuations and scaling laws are in qualitative agreement with experimental observations. The model has severe limitations as compared to a convection experiment[11] and quantitative agreements are therefore not to be expected.

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[9] I am grateful to T. Bohr for suggesting this way of estimating the heat flux.

[10] We have observed an increase in β as $x_B > 0$ ($x_B=0.5$ for instance). Then, however, the boundary layer does not evolve as nicely and $x_B \approx 0$ seems to be a more natural value for the model.

[11] For instance, there is no backflow from the top and there are no conserved quantities.