Hadronic Matrix Elements and the π^+-π^0 Mass Difference

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We estimate the π^+-π^0 electromagnetic mass difference within the $1/N$-expansion approach to hadronic matrix elements. Perturbative QCD and a truncated meson theory describe the high- and low-photon-loop-momentum contributions, respectively. The matching between these complementary pictures for strong interactions is discussed.

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Despite our better understanding of strong interactions, the theoretical study of hadronic weak decays remains a rather difficult problem. For illustration, the $\Delta I=\frac{1}{2}$ rule observed in K decays has defied explanation for more than thirty years. This unsatisfactory situation is due to our present inability of treating strong-interaction corrections to weak processes below the 1-GeV confining scale. The evaluation of the corrections arising from the low loop momenta requires indeed relatively new nonperturbative tools such as the lattice, QCD sum rules, or the $1/N$ expansion.[1] The latter analytical approach is particularly simple and provides already a good *qualitative* description of light-meson strong interactions.[2] After all, the Zweig rule observed in ϕ decays, i.e., at a scale where perturbative QCD breaks down, can only be understood in the framework of the $1/N$ expansion.

It has been recently advocated[3,4] that the $1/N$ expansion also gives *quantitative* predictions for weak hadronic matrix elements. In the large-N limit, the four-quark operators induced by perturbative QCD split into products of two-meson operators. The further nonperturbative corrections arising from physics below 1 GeV can then be estimated within a truncated chiral perturbation theory. Therefore the $1/N$-expansion approach allows a link between the short-distance domain of perturbative QCD and the long-distance domain where a description in terms of hadronic bound states is obviously more appropriate. The difference between this complementary approach and the standard pure chiral-perturbation-theory treatment has been given in Ref. 5.

In this Letter, we consider the π^+-π^0 electromagnetic mass difference using the method developed in Ref. 4. We use the photon momentum to divide the loop integration into two parts. For the photon momentum q^2 between zero and the Euclidean M^2 cutoff, we evaluate the contribution to the mass difference within an effective chiral Lagrangian truncated to the lowest-lying mesons. On the other hand, above M^2, we first use the perturbative-QCD picture to derive an expression in terms of quark field operators and then take the large-N limit to express these operators in terms of meson fields. The advantage of the π^+-π^0 mass difference over the weak hadronic matrix elements treated in Ref. 4 is this unique identification of the loop momenta involved in the QCD and meson pictures.

The leading-order-N (in the $1/N$ expansion) factorizable contribution to the π^+-π^0 electromagnetic mass difference is forbidden by spin and parity. For the same reason, the next-to-leading factorizable contributions are also vanishing. The aim of this Letter is therefore to estimate the next-to-leading nonfactorizable contributions to the π^+-π^0 electromagnetic mass difference.

Let us start with the calculation of the perturbative-QCD contribution above the cutoff M^2. Penguinlike diagrams only induce isospin-zero operators which cancel among each other in the mass difference. Consequently, in the chiral limit, the order-α_s contributions are simply given by the diagrams in Fig. 1. A straightforward calculation of these two Feynman diagrams leads to a (gauge invariant) expression in terms of four-quark current-current and density-density operators. Their π-π hadronic matrix elements are easily estimated using the well-known nonlinear σ model for the pion fields π^a coupled to external vector (axial-vector) currents $\mathcal{V}_\mu$ ($\mathcal{A}_\mu$) and scalar (pseudoscalar) densities $\mathcal{S}(\mathcal{P})$:

$$\mathcal{L}=\tfrac{1}{8}f_\pi^2\,\mathrm{Tr}[\mathcal{D}_\mu U^\dagger \mathcal{D}_\mu U+r(m^\dagger U+U^\dagger m)] \qquad (1)$$

with $m=\mathcal{S}-i\mathcal{P}$, $U=\exp(i\sqrt{2}\pi/f)$, $\pi=\sum_a\lambda_a\pi^a$, and $f_\pi=132$ MeV. The covariant derivative in Eq. (1) is defined by

$$\mathcal{D}_\mu U=\partial_\mu U+i(U\mathcal{V}_\mu-\mathcal{V}_\mu U+U\mathcal{A}_\mu+\mathcal{A}_\mu U)\,. \qquad (2)$$

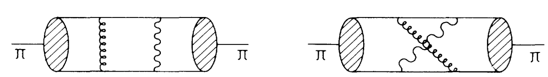

FIG. 1. Nonzero contributions to the short-distance part of the π^+-π^0 mass difference. The wavy line is a photon, the curly line is a gluon, and the full line is a quark.

A straightforward identification with the QCD Lagrangian gives then the following bosonization of currents and densities:

$$\mathcal{V}_\mu^{ji}(\mathcal{A}_\mu) \equiv \bar{q}_j \gamma_\mu(\gamma_5) q_i = i\tfrac{1}{4} f_\pi^2 (\partial_\mu U^\dagger U \pm \partial_\mu U U^\dagger)^{ij}, \qquad (3)$$
$$\mathcal{S}^{ji}(-i\mathcal{P}) \equiv \bar{q}_j (\gamma_5) q_i = -\tfrac{1}{8} f_\pi^2 r (U^\dagger \pm U)^{ij},$$

with $r(\mu) m_s(\mu) \simeq 2 m_K^2$, $m_s(\mu)$ being the strange-quark mass defined at the renormalization point μ.

The large-N approximation ensures the factorization of the four-quark operators induced by the diagrams of Fig. 1. Consequently, only the density-density operators survive in the chiral limit ($m_\pi = 0$) and we obtain the following π^+-π^0 squared mass difference:

$$\Delta m_\pi^2(\text{pert. QCD}) \simeq 6 \frac{\alpha_{\text{em}}}{f_\pi^2} \alpha_s (\langle 0 | q\bar{q} | 0 \rangle)^2 \int \frac{dq^2}{q^4}, \qquad (4)$$

with $\langle 0 | q\bar{q} | 0 \rangle = -f_\pi^2 r/4$. We notice that the result given in Eq. (4) can also be derived in the operator-product-expansion approach where the pion fields are first reduced. The explicit expansion of the induced vector-vector minus axial-axial current operator is given in Ref. 6.

Let us now turn to the evaluation of the meson contributions below M^2. The pseudoscalar exchange contributions [see Fig. 2(a)] are obtained from Eq. (2) with the substitution

$$\mathcal{V}_\mu = e\,\text{diag}(\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}) B_\mu^{\text{em}}, \quad \mathcal{A}_\mu = 0, \qquad (5)$$

and we find

$$\Delta m_\pi^2(0^{-+}) = \frac{3}{4\pi} \alpha_{\text{em}} \int_0^{M^2} dq^2. \qquad (6)$$

If we define the cutoff M to be the (matching) scale where the integrands in Eqs. (4) and (6) are equal, we obtain the reasonable range 0.7 GeV $\geq M \geq$ 0.6 GeV for 0.12 GeV $\leq m_s$ (1 GeV) $\leq$ 0.18 GeV. The total contribution to the π^+-π^0 electromagnetic mass splitting is then twice the short-distance contribution. We obtain $\Delta m \simeq m_{\pi^+} - m_{\pi^0} = 6.4$ (4.3) MeV for $\Lambda_{\text{QCD}} = 0.3$ GeV and $m_s(1\ \text{GeV}) = 0.12$ (0.18) GeV, respectively. For $\Lambda_{\text{QCD}} = 0.2$ GeV, we find $\Delta m = 5.5$ (3.7) MeV. This is in fair agreement with the "observed" mass difference $\Delta m^{\text{expt}} = 4.43 \pm 0.03$ MeV obtained[7] after subtraction of the small effect due to the m_d-m_u quark mass difference.

The severe truncation of the meson theory to the pseudoscalars represents a good first step. However, a comparison between Eqs. (4) and (6) indicates a rather strong dependence on M^2 (see Fig. 3). A direct test of our method is therefore to improve the meson approximation by including heavier resonances.

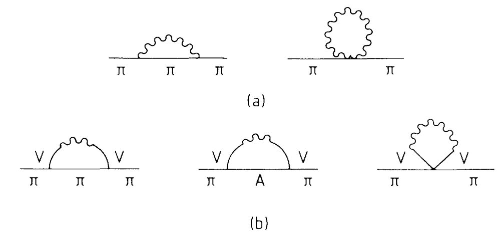

FIG. 2. (a) Feynman diagrams contributing to the long-distance part of the π^+-π^0 mass difference for the low-energy truncation with only pseudoscalar mesons. (b) Same but for the low-energy truncation with the pseudoscalar, vector, and axial-vector mesons.

Let us consider the vector (V_μ) and axial-vector (A_μ) exchange contributions to the π^+-π^0 mass difference [see Fig. 2(b)]. The minimal chiral-invariant "massive Yang-Mills" version of the Lagrangian containing π, ρ, and a_1 mesons and satisfying the two Weinberg sum rules[8] reads[9]

$$\mathcal{L} = \tfrac{1}{8} f^2 \text{Tr} D_\mu U D_\mu U^\dagger - \tfrac{1}{4} \text{Tr}(F_{\mu\nu}^L F_L^{\mu\nu} + F_{\mu\nu}^R F_R^{\mu\nu}) + m_V^2 \text{Tr}[(V_\mu - g^{-1}\mathcal{V}_\mu)^2 + (A_\mu - g^{-1}\mathcal{A}_\mu)^2], \qquad (7)$$

with

$$L_\mu(R_\mu) = V_\mu \mp A_\mu = \tfrac{1}{2}\lambda_a (V_\mu^a \mp A_\mu^a),$$
$$F_{\mu\nu}^{L(R)} = \partial_\mu L(R)_\nu - \partial_\nu L(R)_\mu - ig[L(R)_\mu, L(R)_\nu].$$

The covariant derivative defined with respect to the

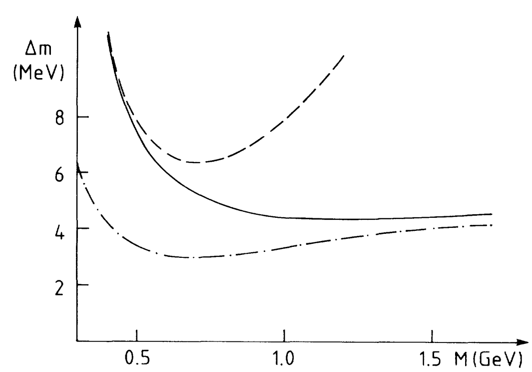

FIG. 3. The π^+-π^0 electromagnetic mass difference as a function of the cutoff M (scale where we match the meson and quark-gluon pictures). Dashed line: long-distance part calculated with pseudoscalar mesons only and with $\Lambda_{\text{QCD}} = 0.3$ GeV, $m_s(1\ \text{GeV}) = 0.12$ GeV. Full line: long-distance part calculated with pseudoscalar, vector, and axial-vector mesons and with $\Lambda_{\text{QCD}} = 0.3$ GeV, $m_s(1\ \text{GeV}) = 0.12$ GeV. Dash-dotted line: same as the full line but with $\Lambda_{\text{QCD}} = 0.2$ GeV and $m_s(1\ \text{GeV}) = 0.18$ GeV.

gauge symmetry is given by

$$D_\mu U = \partial_\mu U - igL_\mu U + igUR_\mu . \quad (8)$$

In the absence of kinetic terms for the (axial-)vector mesons [second term in Eq. (7)], the Lagrangian reduces simply to the kinetic terms given in Eq. (1). Note that the perturbative-QCD result [see Eq. (4)] is not modified since the coupling of the physical pions to the (pseudo)scalar densities remains unchanged. The nondiagonal kinetic term $\partial_\mu \pi A^\mu$ contained in Eq. (7) can be rotated away by the following change of variables:

$$A_\mu \to A_\mu - x\,\partial_\mu \pi, \quad f \to y f_\pi, \quad U \to U , \quad (9)$$

with $x = (m_A^2 - m_V^2)^{1/2}/m_A m_V$, $y = m_A/m_V$, and $m_A^2 = m_V^2 + \frac{1}{2} y^2 f_\pi^2 g^2$. Electromagnetism is incorporated by using the substitution given in Eq. (5). Consequently, vector dominance is implemented and the diagrams of Fig. 2(a) are simply replaced by those of Fig. 2(b), with the momentum-dependent hadronic vertices[10] defined by Eqs. (7) and (9). The integration over the photon low momenta becomes then

$$\Delta m^2(0^{-+},1^{--},1^{++}) = \frac{3}{4\pi}\alpha_{\rm em}\int_0^{M^2} \frac{m_A^2 m_V^2}{(q^2+m_V^2)(q^2+m_A^2)}\,dq^2 . \quad (10)$$

The same result can be derived by using the so-called "hidden-symmetry" approach of Bando, Kugo, and Yamawaki.[11] If we send the cutoff M to infinity in Eq. (10) and assume the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation,[12] i.e., $y = \sqrt{2}$, we recover the famous result obtained[13] using current-algebra techniques; namely, $\Delta m = (3\alpha_{\rm em}/4\pi)(m_\rho^2/m_\pi)\ln 2 \simeq 5.1$ MeV. This successful extrapolation from 1 GeV to infinity requires a huge correlation among heavier resonance contributions.[14] This can only be justified if the large-q^2 contributions are estimated in the framework of perturbative QCD, which is precisely the basic feature of the $1/N$-expansion approach considered.

In the small-q^2 limit ($q^2 \ll m_{V,A}^2$), Eq. (10) simply reduces to Eq. (6). On the other hand, in the large-q^2 limit ($q^2 \gg m_{V,A}^2$), Eq. (10) reproduces the q^2 dependence derived from perturbative QCD and given in Eq. (4). Consequently, the inclusion of the (axial-)vector exchange contributions clearly improves the matching between perturbative QCD valid at large q^2 and the meson picture truncated to the pseudoscalar fields valid at small q^2. This implies a better stability of the total π^+-π^0 mass difference [Eqs. (4) and (10)] with respect to cutoff variations around the 1-GeV scale where both pictures for strong interactions should be reasonable (see Fig. 3). For $m_{a_1} = \sqrt{2} m_\rho$ and $m_s(1\ \text{GeV}) = 120$ (180) MeV, we obtain, respectively,

$$\begin{aligned} \Delta m &= 4.4\ (3.4)\ \text{MeV if } \Lambda_{\rm QCD} = 0.3\ \text{GeV} , \\ \Delta m &= 4.0\ (3.0)\ \text{MeV if } \Lambda_{\rm QCD} = 0.2\ \text{GeV} , \end{aligned} \quad (11)$$

to be compared with $\Delta^{\rm expt} = 4.43 \pm 0.03$ MeV. We note that the numerical results in Eq. (11) are not very sensitive to the value of the a_1 mass.

The explanation of the observed $\Delta I = \frac{1}{2}$ rule in K decays advocated in Ref. 4 is mainly based on the fact that for low loop momenta, the logarithmic operator evolution derived within perturbative QCD is turned into a physical quadratic one, giving rise to sizable long-distance effects despite the small range of integration. Just like for the π^+-π^0 mass difference considered in this Letter, we expect that the (axial-)vector exchange contributions play a crucial role in the matching of the two pictures but not in the numerics. This is in fact supported by an explicit calculation[15] of the effects of vector mesons on another ($\Delta S = 2$) weak hadronic matrix element, i.e., the B parameter.

In the case of the π^+-π^0 mass difference, the identification of the loop momentum of the virtual quarks and gluons with the loop momentum of the virtual mesons is straightforward since they are the same as the one carried by the photon (see Figs. 1 and 2). For weak hadronic matrix elements on the other hand, such an identification is more involved since the standard perturbative-QCD approach requires first the integration of the W propagator. An exact identification would require the conservation of the explicit momentum flowing through the W propagator. However, even in that case, we would be left again with the usual arbitrariness of $\Lambda_{\rm QCD}$ at the one-loop level.

In conclusion, the π^+-π^0 electromagnetic mass difference provides a nice illustration of the $1/N$-expansion approach for hadronic matrix elements proposed in Ref. 4. It emphasizes the crucial role played by the (axial-) vector mesons in the matching of the truncated meson theory and perturbative QCD around 1 GeV. This simple analytical approach to estimate hadronic matrix elements manifestly deserves further investigations and comparisons with other nonperturbative methods.

After completion of the work we became aware of a meson one-loop calculation of the mass difference using a different parametrization for the (axial-)vector mesons.[16]

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