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Direct Determination of the $f(\alpha)$ Singularity Spectrum

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The direct determination of the $f(\alpha)$ singularity spectrum from experimental data is a difficult problem. This Letter introduces a simple method for computing $f(\alpha)$ based on the theorems of Shannon, Eggelston, and Billingsley which is markedly superior to other recently proposed methods, especially when dealing with experimental data from low-dimensional chaotic systems where the underlying dynamics are unknown.

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The long-time behavior of chaotic, nonlinear dynamical systems can often be characterized by fractal or multifractal measures which correspond, for example, to the invariant probability distribution on a strange attractor,[1] the distribution of voltage drops across a random resistor network,[2] the distribution of growth probabilities on the external surface of a diffusion-limited aggregate,[3] or the spatial distribution of dissipative regions in a turbulent flow.[4,5] Various "multifractal formalisms" have recently been developed to describe the statistical properties of these measures in terms of their singularity spectrum[1,6] $f(\alpha)$, or their generalized dimensions[7] D_q.

The $f(\alpha)$ singularity spectrum provides a mathematically precise and naturally intuitive description of the multifractal measure in terms of interwoven sets, with singularity strength α, whose Hausdorff dimension is $f(\alpha)$. If we cover the support of the measure with boxes of size L and define $P_i(L)$ as the probability (integrated measure) in the ith box, then we can define an exponent (singularity strength) α_i by

$$P_i(L) \sim L^{\alpha_i} \tag{1}$$

and, if we count the number of boxes $N(\alpha)$ where the probability P_i has singularity strength between α and $\alpha+d\alpha$, then $f(\alpha)$ can be loosely defined[1] as the fractal dimension of the set of boxes with singularity strength α by

$$N(\alpha) \sim L^{-f(\alpha)}. \tag{2}$$

The "generalized dimensions" D_q, which correspond to scaling exponents for the qth moments of the measure, provide an alternative description of the singular measure.[7] They are defined as

$$D_q = \frac{1}{q-1} \lim_{L\to 0} \frac{\log \sum_i P_i^q(L)}{\log L}. \tag{3}$$

When $f(\alpha)$ and D_q are smooth functions of α and q, then $f(\alpha)$ is simply related to $\tau(q)=(q-1)D_q$ by a Legendre transformation.[1] This relationship reflects a deep connection with the thermodynamic formalism of equilibrium statistical mechanics[8,9] where $\tau(q)$ and q are conjugate thermodynamic variables to $f(\alpha)$ and α. In these cases the $f(\alpha)$ and the D_q curves can be easily transformed into the other. In fact, since the D_q's have in the past been easier to evaluate for measures arising from real or computer experiments, the $f(\alpha)$ curves have usually been determined by the Legendre transform of the $\tau(q)$ curve. Such an operation involves first smoothing the D_q curve and then Legendre transforming. This has several disadvantages. The error bars from the smoothing procedure make the estimation of the error bars from the data itself more difficult. In addition, if the $f(\alpha)$ or $\tau(\alpha)$ curves exhibit any discontinuities, then the smoothing procedure usually causes one to miss these "phase transitions."[10,11]

The purpose of this Letter is to describe a novel procedure for the direct evaluation of $f(\alpha)$ (without resorting to the intermediate Legendre transform), which is mathematically precise and can be readily applied to the analysis of real experimental data where the underlying dynamics are unknown.

Several methods[9,10,12-14] have recently been proposed for the direct computation of $f(\alpha)$ based on log-log plots of the quantities in Eqs. (1) and (2). Unfortunately, the application of these methods to numerical or experimental data suffers from mathematical ambiguities (i.e., is f a Hausdorff or a box dimension) and from large errors due to logarithmic corrections[13,15] which arise from the scale-dependent prefactors in Eq. (2). (Despite these quantitative inaccuracies,[16] these methods do provide important *qualitative* information about the statistical properties of the measure.)

Our method circumvents these difficulties. In an effort to avoid confusion with the other $f(\alpha)$ and D_q formalisms, we will develop this statistical description of the self-similar scaling properties of the singularities of a multifractal measure from first principles.

If we seek to describe a singular measure $P(x)$, then one quantity of interest is the Hausdorff dimension of the measure theoretic support of $P(x)$. This is simply the infimum of the dimensions of the sets on which all the measure lives (i.e., the complements of these sets have zero measure). For a special class of measures that arise from multiplicative processes (described by probabilities P_i), there are several theorems that give us information on how to compute the dimension of the measure theoretic support of such a measure. In particular, we know[17] that the entropy S of such a process is given by

$$S = -\sum_i P_i \log P_i\,, \tag{4}$$

and that the Hausdorff dimension of $\mathcal{M}$, which is the measure theoretic support of the measure associated with such a process, can be related to the entropy by a theorem by Billingsley[18] which gives

$$d_h(\mathcal{M}) = -\lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} P_i \log P_i\,. \tag{5}$$

If we bin the experimental measure under consideration in such a way that the $P_i(L)$ correspond to the probabilities of a multiplicative process with $N \sim L^{-1}$, then Eq. (5) provides a formula for computing the Hausdorff dimension of the set, which is the measure theoretic support of $P(x)$.

We now use these results to evaluate $f(\alpha)$ for a multifractal measure $P(x)$. This is done by first constructing a one-parameter family of normalized measures $\mu(q)$ where the probabilities in the boxes of size L are

$$\mu_i(q,L) = [P_i(L)]^q / \sum_j [P_j(L)]^q\,. \tag{6}$$

As in the definition of the "generalized dimensions" [Eq. (3)], the parameter q provides a microscope for exploring different regions of the singular measure. For $q>1$, $\mu(q)$ amplifies the more singular regions of P, while for $q<1$ it accentuates the less singular regions, and for $q=1$ the measure $\mu(1)$ replicates the original measure. Then the Hausdorff dimension of the measure theoretic support of $\mu(q)$ is given by[19] Eq. (5)

$$f(q) = -\lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} \mu_i(q,L) \log[\mu_i(q,L)] = \lim_{L\to 0} \frac{\sum_i \mu_i(q,L) \log[\mu_i(q,L)]}{\log L}\,. \tag{7}$$

In addition, we can compute the *average value* of the singularity strength $\alpha_i = \log(P_i)/\log L$ with respect to $\mu(q)$ (Ref. 20) by evaluating

$$\alpha(q) = -\lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} \mu_i(q,L) \log[P_i(L)] = \lim_{L\to 0} \frac{\sum_i \mu_i(q,L) \log[P_i(L)]}{\log L}\,. \tag{8}$$

Equations (7) and (8) provide a relationship between a Hausdorff dimension f and an *average* singularity strength α as implicit functions of the parameter q. Moreover, it is easy to exploit the obvious relationship of these definitions of $f(q)$ and $\alpha(q)$ to the definition of the generalized dimensions in Eq. (3) to show[21] that $f = q\alpha - \tau$ and $\alpha = d\tau/dq$. These are precisely the Legendre transform relations[1] between the generalized dimensions and the original singularity spectrum $f(\alpha)$. Therefore, Eqs. (7) and (8) provide an alternative definition of the singularity spectrum, which, most importantly, can be easily used to accurately compute the $f(\alpha)$ curves directly from experimental data without the intermediate Legendre transform of the $\tau(q)$ curve, without confusing the box dimension and the Hausdorff dimension, without neglecting logarithmic corrections, and without suffering from poor sampling statistics for large and small values of the singularity strength. Although the relationships [Eqs. (7) and (8)] have appeared previously in more formal discussions of the relationships between the multifractal and thermodynamic formalisms,[9,21] we believe that we are the first to suggest and demonstrate that these formulas provide a practical, efficient, and highly accurate method for the direct computation of singularity spectrum $f(\alpha)$.

To illustrate our method we examine two different situations. First, we analyze an analytically solvable example (two-scale Cantor measure or binomial measure) to evaluate the accuracy of the method. Then we apply this method to laboratory data for the dissipation field in fully developed turbulence.[4,5]

Consider the two-scale Cantor measure, which is generated by dividing the unit interval into two pieces, each of half the previous length, but with unequal measure (say p_1 and p_2) and repeating this process *ad infinitum*. Then the measure at the nth level of this mulitplicative process would consist of $N = a^n$ pieces (here $a=2$) of equal length, $L = a^{-n}$ with probabilities $P_i(L) = p_1^{n-k} p_2^k$ ($k=0,\ldots,n$). For our example we choose $p_1 = 0.7$ and $p_2 = 0.3$ (Ref. 4) and by construction $a=2$. We calculate the $f(\alpha)$ curve by first covering the measure with boxes of length $L = 2^{-n}$ and computing the probabilities

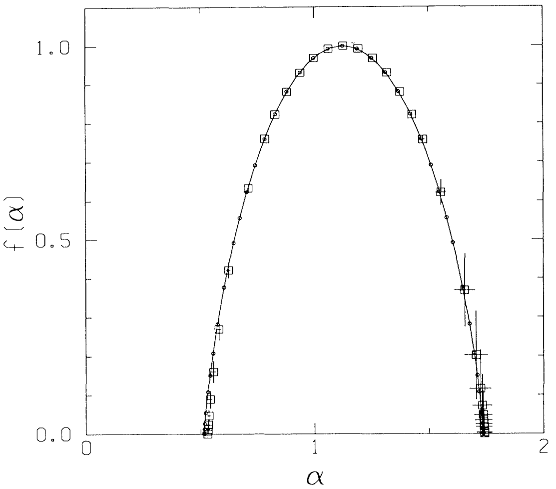

FIG. 1. Comparison of the $f(\alpha)$ curves for a two-scale (binomial) Cantor measure, with $l_1=l_2=0.5$, $p_1=0.7$, $p_2=0.3$, determined analytically (solid line) using our direct method with box sizes of the form 2^{-n} (circles) and $(1.1)^{-n}$ (squares).

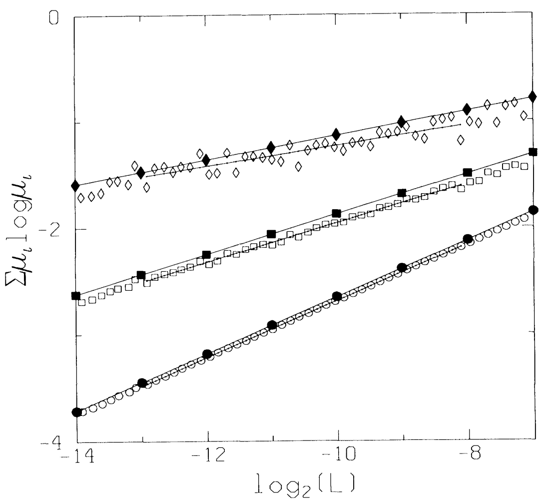

FIG. 2. Examples of linear fits to the semilog plots used to calculate $f(q)$ for the two-scale Cantor measure considered in Fig. 1 with $q=1$ (diamonds), $q=2$ (circles), and $q=3$ (squares). The solid symbols correspond to box sizes of the form 2^{-n} and the open symbols to box sizes of $(1.1)^{-n}$. The error bars in Fig. 1 are determined by the range of reasonable fits to the data.

$P_i(L)$ in each of the boxes.[22] We then construct the one-parameter family of normalized measures with $\mu_i(q,L)$ defined by Eq. (6). Finally, for each value of q we evaluate the numerators on the right-hand sides of Eqs. (7) and (8) $\{\sum_i \mu_i(q,L)\log[\mu_i(q,L)]$ and $\sum_i \mu_i(q,L)\log[P_i(L)]$, respectively$\}$ for decreasing box sizes (increasing n), and we extract $f(q)$ and $\alpha(q)$ from the slopes of the graphs of the numerators versus $\log L$. Figure 1 shows that the $f(q)$ curve calculated from Eqs. (7) and (8) is in excellent agreement with the known analytical result.[1] In Fig. 2, we provide an example of the linear fit to $\sum_i \mu_i(q,L)\log[\mu_i(q,L)]$ vs $\log L$ for three different values of q to show that there is no ambiguity in the determination of the slopes. However, in general (and in most experimental situations) we do not know the correct base a of the multiplicative process. Consequently, the entropy computed using Eq. (7) with some other base will always be greater than, or equal to, the true entropy.[18]

To evaluate our method under these circumstances, we cover the binomial measure with boxes of size $L=(1.1)^{-n}$ and show in Fig. 1 that the agreement of our results for $f(\alpha)$ with the analytical results is still very good. In fact, Billingsley[18] provides bounds on the size of these errors and assures us that in the limit $N\to\infty$ $(L\to 0)$ this result will still converge to the correct Hausdorff dimension. However, Fig. 2 shows that for any finite range of L, we must now find the best linear fit to oscillating points[23] which gives rise to the associated "error bars" in Fig. 1. Notice that despite these errors, our method reproduces the top of the $f(\alpha)$ curve very accurately (including the values of D_0 and D_1) while other methods consistently overshoot or undershoot the values of D_0 and D_1 without carefully including finite-size corrections.[9,10,12–15]

Finally, we apply our method to laboratory data arising from one-dimensional cuts through the dissipation field of fully developed turbulence at a moderate Reynolds number. Earlier work[4,5] has shown that the dissipation field can be characterized by a multifractal. For these one-dimensional cuts, D_0 or $\max[f(\alpha)]$ is equal to 1.0 (Ref. 4). Figure 3 shows a comparison of the $f(\alpha)$

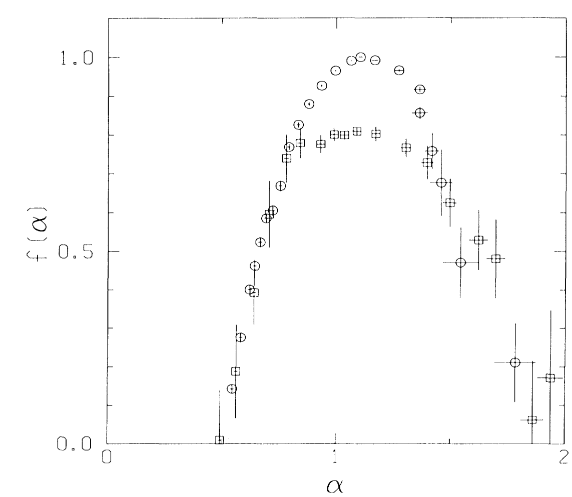

FIG. 3. Comparison between a (microcanonical) method (Ref. 13) (circles) to compute $f(\alpha)$ based on the scaling of histograms (with first-order corrections) and the (canonical) method described here (squares) for the dissipation field of fully developed turbulence at a moderate Reynolds number. From various experimental results (Ref. 4) we know that $\max[f(\alpha)]$ is 1.0.

curves for this data computed from our method and that from the method of scaling of histograms.[13] Note that the direct method proposed in Ref. 13 undershoots the correct value of D_0 by almost 20% despite taking first-order corrections into account.[24]

In conclusion, we have proposed a simple yet accurate method for the direct determination of $f(\alpha)$ that is specially suited for analyzing experimental data. It is always extremely accurate in the region around D_0, whether or not we know the underlying dynamics. This is important because it is precisely in this region that experimental data are the most reliable and one needs a method to process them without adding any errors. A detailed discussion on the calculation of $f(\alpha)$ curves for fully developed turbulence[24] and for maps displaying phase-transition behavior[25] will be published shortly.

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[1]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[2]L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **34**, 4656 (1986).

[3]P. Meakin, A. Coniglio, H. E. Stanley, and T. Witten, Phys. Rev. A. **34**, 3325 (1986).

[4]C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).

[5]R. R. Prasad, C. Meneveau, and K. R. Sreenivasan, Phys. Rev. Lett. **61**, 74 (1988).

[6]U. Frisch and G. Parisi, in *Turbulence and Predictability of Geophysical Flows and Climate Dynamics,* International School of Physics "Enrico Fermi," Course LXXXVII, edited by M. Ghill, R. Benzi, and G. Parisi (North-Holland, New York, 1985).

[7]H. G. E. Hentschel and I. Procaccia, Physica (Amsterdam) **8D**, 435 (1983).

[8]M. J. Feigenbaum, J. Stat. Phys. **46**, 919 (1987).

[9]B. B. Mandelbrot in *Fluctuations and Pattern Formation, Cargèse,* edited by H. E. Stanley and N. Ostrowsky (Kluwer, Dordrecht, 1988).

[10]P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988).

[11]Since $f(\alpha)$ and $\tau(q)$ correspond to thermodynamic quantities, these discontinuities are analogous to phase transitions.

[12]A. Arneodo, G. Grasseau, and E. J. Kostelich, Phys. Lett. A **124**, 424 (1987).

[13]C. Meneveau and K. R. Sreenivasan (to be published).

[14]R. Badii and G. Broggi, Phys. Lett. A **131**, 131 (1988).

[15]W. van der Water and P. Schram, Phys. Rev. A **37**, 3118 (1988).

[16]Mandelbrot notes (Ref. 9) that for a binomial measure, direct application of Eqs. (1) and (2) results in an overshoot from the true value of D_0 which is visible even after using over 10^{15} intervals. Similar overshoots can be observed in Refs. 12–14.

[17]C. Shannon, Bell Systems Tech. J., **27**, 379–423 (1948); **27**, 623–656 (1948).

[18]P. Billingsley, in *Ergodic Theory and Information* (Wiley, New York, 1965).

[19]In this formal argument we have assumed that the limiting $\mu(q)$ also corresponds to a multiplicative process as is true of the two-scale Cantor measure.

[20]From the thermodynamic point of view (Feigenbaum, Ref. 8) our method can be understood as a canonical way of computing $f(\alpha)$, while other methods of direct determination have so far been microcanonical.

[21]M. H. Jensen, L. P. Kadanoff, and I. Procaccia, Phys. Rev. A **36**, 1409 (1987).

[22]When D_0 is less than one, box counting using boxes of equal size often results in the inaccurate determination of the right-half of the $f(\alpha)$ curve.

[23]L. A. Smith, J. D. Fournier, and E. A. Spiegel, Phys. Lett. **114A**, 465 (1986).

[24]Chhabra, C. Meneveau, R. V. Jensen, and K. R. Sreemivasan (to be published).

[25]A. Chhabra and R. V. Jensen (to be published).