Relativistic Bursts

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We obtain an exact solution for a relativistic Langmuir wave with nontrivial space and time dependence. We find explosive behavior in the electron density and hence also in the electric field gradient. Relativistic effects are thus found to be a possible candidate to explain this interesting type of behavior, recently found in numerical simulations (though it is too early to give a quantitative treatment).

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The nonrelativistic problem of nonlinear electron oscillations in a cold plasma was solved by several people independently some time ago.[1-4] Here we will refer to the solution as given in Davidson's book,[4] though this Letter is self-contained. The method of Refs. 3 and 4 was to transform to Lagrangian coordinates which follow the fluid motion (for extensions see Ref. 5). Any initial periodic profile with wavelength $2\pi/k$ could lead to a well defined oscillation, provided that the electric field E did not exceed a critical value of order $m_e\omega_{pe}^2/ek$, where ω_{pe} is the electron plasma frequency. If the critical value was taken, the density eventually became infinite at a point. More recently, the present authors were able to solve the same problem for a warm electron, cold ion plasma.[6] The result involved a drastic reduction in the class of possible initial conditions that would lead to a coherent oscillation. Thus, addition of even a small electron temperature constitutes a singular perturbation. However, the dependence of $E_{\max}$ on the temperature was found to be weak. Traveling Berstein-Greene-Kruskal (BGK) modes, which depend on just one variable, $x-Vt$, have also been extensively investigated for both cases.[7-11] Early work by Soviet scientists should be mentioned in this context.[12]

The availability of high-powered lasers and various accelerator concepts based on laser-plasma interaction has renewed interest in strongly nonlinear waves. In particular, experimenters wish to know whether wave breaking is to be expected and, if so, what will be the largest possible value of E when this happens in a particular experimental setup.[12-18] Lagrangian methods are sometimes used in this context.[12-14]

In this Letter we give a relativistic treatment of the cold plasma problem and somewhat unexpectedly find explosive behavior; that is, for sufficiently long time the density becomes infinite.

The equations governing a cold, relativistic electron plasma in which the ions constitute a uniform background are, in one space dimension,

$$\partial n/\partial t+\partial(nv)/\partial x=0\,, \tag{1}$$

$$dp/dt=-eE/m\,,\quad p=P/m=v/(1-v^2/c^2)^{1/2}\,,$$
$$d/dt=\partial/\partial t+v\,\partial/\partial x\,, \tag{2}$$

$$\partial E/\partial t=4\pi nev\,, \tag{3}$$

$$\partial E/\partial x=-4\pi e(n-n_0)\,. \tag{4}$$

Here the momentum P, density n, and velocity v are all electron quantities, and n_0 is the constant ion density. E is the electric field intensity. Combining the Maxwell and Poisson equations, (3) and (4), we obtain

$$dE/dt=4\pi n_0 v\,. \tag{5}$$

We now introduce Lagrangian coordinates x_0,τ which follow the fluid such that $x_0=$const gives the motion of one fluid element:

$$x_0=x-\int_0^\tau v(x_0,\tau)d\tau,\quad \tau=t\,. \tag{6}$$

The identities

$$d/dt=\partial/\partial\tau\,,\quad \partial/\partial x=D^{-1}\partial/\partial x_0\,,$$
$$D=\partial x/\partial x_0=1+\int_0^\tau(\partial v/\partial x_0)d\tau\,, \tag{7}$$

will be needed in what follows. Equation (1) now yields

$$n(x_0,\tau)=n(x_0,0)/D\,. \tag{8}$$

Differentiation of (2), expressed in the new variables, together with (3) yields

$$\partial^2p/\partial\tau^2+\omega_{pe}^2p/v(1+p^2/c^2)^{1/2}=0\,. \tag{9}$$

This equation was first derived by Polovin.[12] This is the equation of a relativistic harmonic oscillator. It is easily solved to give τ in terms of p. Integrating once, we obtain

$$\partial p/\partial \tau = \pm \omega_{pe}[2a-(1+p^2/c^2)^{1/2}]^{1/2}, \tag{10}$$

where a is a function of x_0. We now introduce α and r through

$$\sin^2\alpha = \frac{a-(1+p^2/c^2)^{/12}}{a-1}, \quad r^2=(a-1)/(a+1), \quad \alpha(0)=\pi/2, \tag{11}$$

to obtain, via a second integration,

$$\omega_{pe}\tau = [2(a+1)]^{1/2}E(\alpha,r) - [2/(a+1)]^{1/2}F(\alpha,r)+\phi(x_0). \tag{12}$$

Here E and F are incomplete elliptic integrals, and ϕ is a second arbitrary function of x_0.

Although a solution to (1)-(4) can now be found for any periodic initial conditions, here we specify them to correspond to the conditions chosen for the nonrelativistic case.[4] The relativistic form is

$$n(x_0,0)=1+\Delta\cos(kx_0), \quad \Delta<1, \qquad p(x_0,0)=0, \tag{13}$$

where Δ is a dimensionless constant. Now $\phi(x_0)$ is a sum of two complete elliptic integrals such that

$$\omega_{pe}\tau = [2(a+1)]^{1/2}[E(\alpha,r)-E(r)]-[2/(a+1)]^{1/2}[F(\alpha,r)-K(r)]. \tag{14}$$

A second step is to use the arguments of the elliptic functions, α and r, as the basic parameters. This leads to an unceremonious elimination of the Lagrangian variables and gives $n(x,t)$ in parametric form. Using (8), (13), and (14), we obtain

$$n=\frac{n_0\{1\pm\Delta[1-4r^2/\Delta'^2(1-r^2)]^{1/2}\}}{1\pm\Delta[1-4r^2/\Delta'^2(1-r^2)]^{1/2}(1-\sin\alpha-A\cos\alpha)}, \tag{15}$$

$$A=\frac{\{(1+r^2)[E(\alpha,r)-E(r)]-(1-r^2)[F(\alpha,r)-K(r)]\}(1-r^2\sin^2\alpha)^{1/2}+[r^2(1-r^2)/2]\sin(2\alpha)}{2r^2\sin^2\alpha-1-r^2}, \tag{16}$$

$$kx=\arcsin[\pm 2r/\Delta'(1-r^2)^{1/2}]\pm 2r[\Delta/\Delta'(1-r^2)^{1/2}](1-\sin\alpha), \tag{17}$$

$$\omega_{pe}t=[2/(1-r^2)^{1/2}][E(\alpha,r)-E(r)]-(1-r^2)^{1/2}[F(\alpha,r)-K(r)], \tag{18}$$

$$\Delta'=\omega_{pe}\Delta/ck, \quad 1>\Delta'(4+\Delta'^2)^{1/2}\geq r\geq 0, \quad \alpha\geq\pi/2.$$

This is an exact solution in parametric form $n=n(\alpha,r)$, $x=x(\alpha,r)$, $t=t(\alpha,r)$, and essentially specified by the dimensionless numbers Δ and ω_{pe}/ck. It is of some mathematical interest, being a fully x,t dependent solution, that the set of equations may not be integrable by inverse scattering.

As the integrals $E(\alpha,r)$ and $F(\alpha,r)$ in A are extended beyond $\pi/2$, secular behavior is observed and the denominator will vanish after finite time (finite α). However, as we are not used to elliptic functions extended beyond $\pi/2$, it takes a moments reflection to see this secular behavior. The salient features can be seen, for example, by looking at the weakly relativistic limit, accurate up to and including terms of order ω_{pe}^2/k^2c^2. [Nevertheless, for some problems the full solution (15)-(18) would have to be used. An example is the beat-wave application for which ω_{pe}/kc is close to unity; see remarks at end of this Letter.] It should be stressed that, equipped with the exact solution, we can cope with all possible values of ω_{pe}/kc, the weakly relativistic limit simply furnishing an illustration.

We revert to Lagrangian variables for this limit. The result is

$$n=\frac{n_0[1+\Delta\cos(kx_0)]}{1+\Delta\cos(kx_0)[1-\cos(\tilde{\omega}_{pe}t)-\frac{3}{8}(\Delta\omega_{pe}/kc)^2\omega_{pe}t\sin^2(kx_0)\sin(\tilde{\omega}_{pe}t)]}, \tag{19}$$

$$\tilde{\omega}_{pe}=\omega_{pe}[1-\tfrac{3}{16}(\Delta\omega_{pe}/kc)^2\sin^2(kx_0)], \tag{20}$$

$$kx=kx_0+2\Delta\sin^2(\tilde{\omega}_{pe}t/2)\sin(kx_0), \quad t\geq 0, \quad \Delta\leq\tfrac{1}{2}. \tag{21}$$

For $kx_0=n\pi/2$ this approximation is, in fact, exact. The frequency shift (20) agrees with earlier values obtained by different methods when $\sin^2 kx_0$ takes its maximum value. The solution (19)-(21) should be compared with Eqs. (28)-(30) on p. 38 of Davidson's book[4] for the nonrelativistic limit, $c\to\infty$. Relativity brings in a dependence of the frequency on the amplitude and on x_0, and a secular term in the denominator of n. Thus the "oscillation" is no longer periodic. If we keep x_0 constant, we can follow an individual fluid element, and see how it moves and condenses as t increases. Unless kx_0 is a multiple of $\pi/2$, the density will eventually become infinite due to the secular term in the denominator (for a somewhat similar situation in a different problem see Ref. 19). Our solution (19)-(21) can only really be used up to and including the first flare-up of this kind. This will happen for a specific value of x_0 and after roughly $(kc/\Delta\omega_{pe})^2$ "oscillations" when this number is large. The exact value of t corresponding to this flare-up can be obtained by demanding that the denominator of n be zero and x_0 its first root.

It is important to note that this explosive behavior will be observed for general initial conditions. To see this, note from (9) that since $v=p/(1+p^2/c^2)^{1/2}$,

$$D=1-\omega_{pe}^{-2}\frac{\partial}{\partial x_0}\left[\frac{\partial p}{\partial \tau}-\frac{\partial p}{\partial \tau}\bigg|_0\right].$$

Since the solutions to (9) can be written in the form $p=p[\tau/T(x_0),x_0]$ where T is the period, differentiation leads to a secular contribution to D proportional to $\tau/T^2 dT/dx_0$. This term will cause D to vanish after a finite τ, giving an infinite value for n.

An exception to the above explosive behavior is furnished by the BGK waves of Akhiezer and Polovin[12]; see also the recent paper by Katsouleas and Mori.[20] These structures are functions of just one variable $x-vt$ and do not exhibit explosive behavior. However, the initial conditions required to set them up are very special and would involve nonzero $p(x_0,0)$ in contradistinction to (19)-(21). (For more on this topic for nonrelativistic plasma waves, see Albritton and Rowlands,[8] where these very special initial conditions are given explicitly. Extension to the relativistic case would not be difficult.)

All other initial conditions, including all such that $p(x_0,0)=0$, lead to our relativistic bursts.

In a recent simulation,[21] a large amplitude, relativistic Langmuir wave was studied numerically. The wave was found to steepen much more severely than expected. We suggest that this steepening might be due to the secular behavior of the denominator in the expression for the density; for example, as given by (19) [in the simulations $(\Delta\omega_p/kc)^2=0.16$]. This would then be due to relativistic effects and hence it would not be surprising that it was not found in the nonrelativistic simulation. However, there is not enough detail in Ref. 21 for a fit between our theory and the simulation. We put this idea forward as worthy of further consideration and not as a definite proof. It should be stressed that our new exact solution describes a wide range of situations and should be the source of many comparisons with results of simulations in the future.

In conclusion, it has proven possible to explain the unexpected steepening of the electric field of a relativistic Langmuir wave found in numerical simulations[21] by looking at the corresponding exact solution. As the crucial behavior is explosive (the denominator vanishes), a straightforward expansion in $1/c^2$ or Δ would not be sufficient to obtain this result. In practice, infinite density cannot of course exist and the basic model given by (1)-(4) must be extended to include more physics. This would limit the density excursion but presumably after the first burst the present model could be applicable once again. The fact that the occurrence of a burst is inevitable in our model, almost regardless of initial conditions, suggests that in a real system a succession of bursts will occur.

It is not clear what the extra physics referred to above should be. One's first thought would be to include thermal effects as was done for nonrelativistic fluid plasmas in Ref. 6 and for relativistic BGK waves in Ref. 20. However, as stressed above, BGK waves are a very special case not exhibiting bursts, whereas according to Ref. 6, thermal effects in the fluid model limit the class of possible initial conditions but do not preclude those leading to relativistic bursts. Thus one should probably look further afield, perhaps to Landau damping. However, this would not be covered by any fluid model.

The question of the relavence to the beat-wave application is open at this stage. The present theory does not cover it, but could perhaps be so extended. The simple analysis of Akhiezer and Polovin, which is a BGK mode treatment, may be more relevant. This would depend on whether bursts are observed in beat-wave applications. We leave this problem to a future paper.

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[1]J. M. Dawson, Phys. Rev. **113**, 383 (1959).

[2]O. Buneman, Phys. Rev. **115**, 503 (1959).

[3]R. C. Davidson and P. P. Schram, Nucl. Fusion **8**, 183 (1968).

[4]R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York, 1972).

[5]E. Infeld and G. Rowlands, "A Lagrangian Approach to Nonlinear Plasma Dynamics" (to be published).

[6]E. Infeld and G. Rowlands, Phys. Rev. Lett. **58**, 2013 (1987).

[7]R. C. Davidson, Phys. Fluids **11**, 194 (1968).

[8]J. Albritton and G. Rowlands, Nucl. Fusion **15**, 1199 (1975).

[9]T. Speziale and P. J. Catto, Phys. Fluids **20**, 990 (1977).

[10]E. Infeld and G. Rowlands, J. Phys. A **12**, 2255 (1979).

[11]W. L. Kruer, Phys. Fluids **22**, 1111 (1979).

[12]A. I. Akhiezer and G. Y. Lyubarski, Dokl. Akad. Nauk SSSR **53**, 193 (1951); A. I. Akhiezer and R. V. Polovin, Dokl. Akad. Nauk SSSR **102**, 919 (1955); Zh. Eksp. Teor. Fiz. **31**, 354 (1956); R. V. Polovin, Sov. Phys. JETP **4**, 290 (1957).

[13]M. N. Rosenbluth and C. S. Liu, Phys. Rev. Lett. **29**, 701 (1972); P. S. Lee, Y. C. Lee, C. T. Chang, and D. S. Chun, Phys. Rev. Lett. **30**, 538 (1973).

[14]C. M. Tang, P. Sprangle, and R. N. Sudan, Phys. Fluids

28, 1974 (1985).
[15]J. T. Mendoça, J. Plasma Phys. **34**, 115 (1985).
[16]R. J. Noble, Phys. Rev. A **32**, 460 (1985).
[17]C. J. McKinstrie and J. M. Kindel, in "Applications of Laser Plasmas," edited by L. J. Radziemski and D. A. Cremers (Dekker, New York, to be published).
[18]C. Joshi, W. B. Mori, T. Karsouleas, J. M. Dawson, J. M. Kindel, and D. W. Forslund, Nature (London) **311**, 525 (1984).
[19]E. Ott and C. T. Dunn, Phys. Fluids **14**, 959 (1971).
[20]T. Katsouleas and W. B. Mori, Phys. Rev. Lett. **61**, 90 (1981).
[21]C. J. McKinstrie and D. W. Forslund, Phys. Fluids **30**, 904 (1987).