PHYSICAL REVIEW LETTERS

VOLUME 62 — 6 MARCH 1989 — NUMBER 10

Scaling and Fractal Dimension of Ising Clusters at the $d=2$ Critical Point

A. L. Stella

Dipartimento di Fisica and Centro Interuniversitario di Struttura della Materia (CISM), Università di Padova, I-35131 Padova, Italy, and Scuola Internazionale Superiore di Studi Avanzati (ISAS), I-34100 Trieste, Italy

C. Vanderzande

Limburgs Universitair Centrum, B-3610 Diepenbeek, Belgium

(Received 15 September 1988)

Using formal arguments based on conformal invariance and on the connection between correlated-site percolation and the q-state Potts model with vacancies, we show that the exponents describing Ising clusters at Onsager's critical point are those of the tricritical $q=1$ Potts model. This implies, in particular, a fractal dimension $\bar{d}=\frac{187}{96}$ and a percolative susceptibility exponent $\gamma=\frac{91}{48}$, in good agreement with existing numerical estimates. This $\bar{d}$ is also clearly supported by a new very accurate Monte Carlo finite-size scaling determination. We also conjecture an exponent $y_J=\frac{13}{24}$ controlling the crossover between clusters and droplets.

PACS numbers: 05.50.+q, 05.70.Jk, 64.60.Ak

The study of connected clusters of sites with, e.g., up spin in the Ising model has a long history, which goes back to the formulation of the phase transition problem in terms of a droplet model.[1] Many investigations[2-12] tried to clarify the connections between the percolation properties of such clusters and standard, magnetic properties of the Ising model. In spite of many attempts, however, a full and exact characterization of the scaling properties of Ising clusters, e.g., at the $d=2$ Onsager's critical point, is missing at the moment. In particular, for these clusters, the fractal dimension $\bar{d}$ is still controversial, and only relatively crude numerical estimates exist for it.[12]

In this Letter we present theoretical arguments and numerical results which fully elucidate the nature of the $d=2$ Ising critical point as a multicritical point for correlated-site percolation.

Let us consider, e.g., the Ising model on the square lattice, with reduced Hamiltonian

$$-\beta H(\{S\})=K\sum_{\langle ij\rangle}S_iS_j+h\sum_i S_i\,, \tag{1}$$

where the first sum is over nearest-neighbor (nn) sites, and $S_i=\pm 1$. On the basis of rigorous work, we know that in the (h,K) plane there exists an infinite cluster of spins in the whole region $h>0$, $K\geq K_c=\frac{1}{2}\ln(1+\sqrt{2})$.[5] In addition, an infinite cluster of up spins is present in a region $h>h_0(K)$, $K<K_c$. While the function $h_0(K)$ is not known, some of its properties are. $h_0(K_c)=0$, and, as K goes to zero, $h_0(K)$ should approach a positive value $h_0(0)$, which is related to the site percolation threshold probability of the square lattice $p_c\simeq 0.59$[13] by the equation $\exp[h_0(0)]/2\cosh h_0(0)=p_c$. It has been shown rigorously that Onsager's critical point ($h=0$, $K=K_c$) is also critical with respect to percolation of up spins.[5] In addition, the whole line $K<K_c$, $h=h_0(K)$ is expected to be critical. The point $K=0$, $h=h_0(0)$ corresponds to standard, uncorrelated-site percolation. For $K>K_c$, it has been shown that the percolative order parameter is zero for $h=0^-$ and >0 for $h=0^+$.[5] This implies that $h=0$, $K>K_c$ corresponds to a first-order transition line for the correlated percolation problem. Thus, it is natural to expect that the Ising critical point should have the nature of a tricritical point as far as percolated properties are concerned. This fact is suggested very clearly, e.g., by a recent numerical investigation of interacting-site percolation on the square lattice.[14] In this study a phenomenological renormalization-group

(RG) analysis of the problem gives clear evidence of a tricritical point, located at parameter values which are in very reasonable agreement with Onsager's exact critical values ($K \simeq 0.45$ instead of 0.44... and $h \simeq 0.02$ instead of 0). In Ref. 14 an estimate of the tricritical ν exponent is also reported ($\nu \simeq 0.513$). However, while there this value is interpreted as an indication that the problem could belong to the same universality class as the Θ point of branched polymers,[15] here we will argue differently.

We consider a dilute q-state Potts model on a square lattice with the reduced Hamiltonian:

$$-\beta H(\{n\},\{\sigma\}) = L\sum_{\langle ij\rangle} n_i n_j + \Delta\sum_i n_i + J\sum_{\langle ij\rangle} n_i n_j(\delta_{\sigma_i,\sigma_j}-1) + H\sum_i n_i(\delta_{\sigma_i,1}-1)\,, \quad (2)$$

where $n_i = 0,1$ is a lattice-gas variable, and $\sigma_i = 1,2,\ldots,q$ is a Potts variable. The first two terms of (2) reduce to the Ising Hamiltonian (1) if we put $n_i = (1+S_i)/2$ and $L = 4K$, $\Delta = 2(h-K)$. For $q < 4$ this problem is expected to have both critical and tricritical points. For the tricritical branch, in particular, Coulomb-gas methods and conformal invariance results give analytic expressions for the exponents as a function of q.[16,17]

By standard methods one can show that the partition function of problem (2) on an N-site box becomes[7,18]

$$Z = \mathrm{Tr}_{\{n\}} \exp\left[L\sum_{\langle ij\rangle} n_i n_j + \Delta\sum_i n_i\right] q^{N-\sum_i n_i} \times \mathrm{Tr}_\gamma p_B^{R_\gamma}(1-p_B)^{L_\gamma}\prod_C[1+(q-1)e^{-Hn_c}]\,, \quad (3)$$

where the trace over σ's has been performed and $p_B = 1-\exp(-J)$, can be interpreted as the probability that a bond is present between nn sites that are occupied ($n=1$) in a lattice-gas configuration $\{n\}$. γ indicates bond configurations and R_γ and L_γ are the numbers of bonds, respectively, present or missing in the lattice restricted to the occupied sites. The product runs over clusters of occupied sites which are connected in terms of occupied bonds, and n_c denotes the number of sites in cluster C. This partition function, or the corresponding free energy, $f = \lim_{N\to\infty}\ln(Z/N)$, has two important properties. For $q \to 1$ and $H=0$, f reduces to the Ising lattice-gas free energy, since the trace over γ in Eq. (3) gives 1 for every $\{n\}$. Actually this is also true for $H \neq 0$ because f becomes independent of H in the $q \to 1$ limit, with the particular subtraction we embodied in the H term of Eq. (2). For $H=0$, however, the above property would still hold also without such subtraction. Thus in the $q \to 1$ limit the free energy obtained from (4) is independent of J and always equal to the corresponding lattice-gas free energy along straight lines at constant L and Δ in (J,L,Δ) space. However, one cannot yet draw thermodynamic consequences. The free energy f, even if relevant for conformal invariance considerations (see below), is not the appropriate generating function for percolative quantities. Such a generating function is instead given by $g = \partial f/\partial q\,|_{q=1}$. The first and second derivatives of g with respect to H, at $H=0$, are indeed simply related to the percolative order parameter and to the mean cluster size, respectively, of site-bond correlated percolation.[7,18] In this problem two nn up-spin sites in an Ising configuration are considered as connected with probability p_B. In the limit $J \to \infty$ ($p_B = 1$) we recover, in particular, correlated-site percolation.

For a two-dimensional scale-invariant system which is also conformal invariant, it has been shown that, when the model is defined on an $M\times\infty$ periodic strip, the free energy per site f_M approaches the thermodynamic limit f according to the asymptotic law[19]

$$f_M = f - \pi c/6M^2 + O(1/M^3)\,, \quad (4)$$

where c is the central charge of the theory, the key number for a determination of all critical exponents within the conformal classification scheme.[17] From conformal invariance it is known that both the tricritical point in the $q=1$ Potts model and the Ising critical point have $c = \frac{1}{2}$.[17]

It is now crucial to remark that the behavior of (4), for $c \neq 0$, is specific of scale-invariant points. The approach would be exponential in M, if the point were not critical. As the free energy of the tricritical $q=1$ Potts model (3) only depends on Δ and L and equals that of the lattice gas, and as the critical Ising point ($\Delta_c = 3.52\ldots$, $L_c = 1.76\ldots$, or $K = K_c$, $h = 0$ for the square lattice) is the only scale-invariant point of the lattice gas for which Eq. (4) holds with $c = \frac{1}{2}$, the tricritical points of the $q=1$ Potts model must lie along the line $h=0$, $K = K_c$.

Even though the free energy given by the logarithm of (3) does not have the properties of a true generating function, it turns out that the Kac table[18] associated with the central charge obtained on the basis of (4) from that free energy contains the right critical exponents. This is, e.g., also the case for ordinary percolation and for the problem of self-avoiding walks.[17,20,21]

Conformal invariance combined with other arguments allows us to establish a number of properties of the RG flow on and around the line $K = K_c$, $h = 0$. First of all this line must be invariant and fully unstable for the RG flow. Indeed $K = K_c$, $h = 0$, according to what we argued above, is the only locus of scale-invariant points with $c = \frac{1}{2}$ in (J^{-1},h,K) space. In this space the $q=1$ Potts tricritical fixed point (f.p.) is repulsive in only two directions[16]; as a consequence it must be attractive along the line. If it were not so, the existence of scale-invariant points with $c = \frac{1}{2}$ out of the line would follow.

A case in which the properties of model (2) become transparent is $J=0$. For every q, and for $J=0$, the model reduces to the standard Ising model. So we expect that the point $J^{-1} = \infty$, $h=0$, $K = K_c$ has the critical behavior of the Ising model in $K = K_c$ and h, with exponents equal to 1 and $\frac{15}{8}$, respectively. Since $J=0$ should be transformed into $J'=0$ under renormalization, we also expect $J^{-1}=\infty$, $h=0$, $K=K_c$, to be a f.p., which has to be attractive in view of the Ising character and of the already mentioned relevance in $K-K_c$ and h.

So, on our line the region with high J^{-1} is in the domain of attraction of this f.p. Compatibility with the existence of the attractive $q=1$ Potts tricritical f.p. clearly requires at least one further fully repulsive f.p. on the line at some finite J. This f.p. must be located at the special symmetry point with $J=L/2=2K$. Indeed, as noticed in Ref. 7, for $J=2K$, and $H=0$, model (2) becomes equivalent to an asymmetric $(q+1)$-state Potts model. When $q=1$ the model reduces to an Ising model. So, for $q=1$ we must have a multitricritical f.p. with $c=\frac{1}{2}$ at $J=2K_c$, and the critical singularities must be those of the Ising model in $K-K_c$ and h. The RG flow consistent with the above properties is sketched in Fig. 1. Besides being the most simply conceivable, it is also clearly supported by approximate RG flow diagrams obtained in Ref. 7. The flow pattern in Fig. 1 is also consistent with the flows one can obtain for $q=2$ and $q=3$ realizations of model (2).[22]

Thus we conclude that the point ($J^{-1}=0$, $h=0$, $K=K_c$) is attracted by the tricritical $q=1$ Potts f.p.

There are four $q=1$ Potts tricritical exponents[16,17]: two magnetic, of which the leading one is $y_H=\frac{187}{96}$, associated to the field H in Eq. (2), and two thermal ones, $y_t=\frac{15}{8}$ and $y=1$, which should describe the scaling flow when we move, at $H=0$, in the neighborhood of $h=0$, $K=K_c$. As far as the last two exponents are concerned, it should be noticed that these values were already conjectured for Ising clusters on the basis of Migdal calculations.[7] Here we combine these values with a precise prediction also for y_H, which, in a percolation problem is expected to yield the fractal dimension $\bar{d}$ of the clusters.[23,24] This fractal dimension has been investigated recently by Monte Carlo methods and estimated as $\bar{d}=1.90\pm0.006$,[12] which is quite compatible with our result, $\bar{d}=1.947\ldots$ Our predictions have a more sharp agreement with another, presumably more accurate numerical result, obtained by analysis of high-temperature expansions of the percolative susceptibility of Ising clusters, for $K\to K_c$, and $h=0$.[25] The corresponding exponent was estimated as $\gamma=1.91\pm0.01$. According to our conjecture $\gamma=(2y_H-2)/y=\frac{91}{48}=1.895\ldots$, which is in excellent agreement with the numerical result. When we move towards K_c at $h=0$ in the (h,K) plane, as one can argue using scaling-field arguments, the appropriate exponent under which $K-K_c$ has to be rescaled is indeed y as given above.

Since the direct $\bar{d}$ determination of Ref. 12 is not very precise, in order to have compelling numerical evidence of the correctness of our predictions, we made an independent effort to obtain y_H, an exponent which was not studied before. For Ising models defined on periodic squares with side L $(4\leq L\leq 36)$ and with interactions fixed at Onsager's critical values we estimated the percolative susceptibility very accurately by extensive Monte Carlo runs. Such a quantity is expected to scale like L^{2y_H-d} and thus allows a direct determination of y_H. With high confidence we estimated $y_H=1.94\pm0.01$ which is very consistent with our conjecture.[26]

The ν exponent obtained in Ref. 14 is nothing but an estimate of y_t^{-1}. The value $\nu=0.53\ldots$ following from our arguments is certainly not contradicted by this numerical result, which, unlike in the case of our y_H determination, relies on an approximate location of the tricritical point.

Our results, combined with other known properties, provide insight about the shape of the critical boundary $h_0(K)$ in the neighborhood of Onsager's critical point. We know that the line $h=0$, $K>K_c$ in the $J^{-1}=0$ plane is a line of discontinuous transitions. We thus expect it to belong to a whole surface of first-order transition points in (J^{-1},h,K) space, which should be left invariant under renormalization transformations applied to site-bond correlated percolation. This surface is expected to end at the line $h=0$, $K=K_c$, where it meets with a second-order transitions surface whose intersection with the $J^{-1}=0$ plane is given by the equation $h=h_0(K)$ introduced above. By introducing suitable scaling fields corresponding to y_t and y, respectively, and by taking into account the first-order transition for $h=0$ and $K>K_c$, it is easy to show that

$$h_0(K)\propto(K_c-K)^{15/8} \quad (5)$$

for $K\to K_c^-$. Thus the curve $h=h_0(K)$ has zero slope at $K=K_c$, i.e., the critical percolation line, at the tricriti-

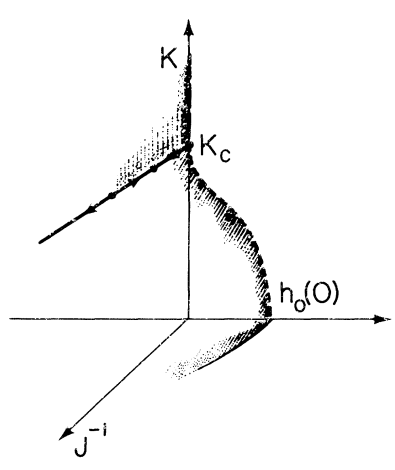

FIG. 1. Qualitative picture in (J^{-1},h,K) space. The short-dashed curve represents $h=h_0(K)$. The straight line $K=K_c$, $h=0$ is a line of scaling invariant points with central charge $c=\frac{1}{2}$. The fully repulsive fixed point on this line is predicted in Ref. 7 and corresponds to Ising droplets. The tricritical $q=1$ Potts f.p. is attractive on the line. On the K axis for $K>K_c$ (long-dashed line) we have first-order transitions.

cal point, has the same slope as the first-order transition line. This is qualitatively shown in Fig. 1.

Conformal invariance allows further insight into the properties of model (2) for $J^{-1}\neq 0$. The above-mentioned unstable f.p. at $J=2K_c$ must have exponents consistent with the Kac table[18] with $c=\frac{1}{2}$, from which both the Ising critical and the Potts $q=1$ tricritical indices can also be derived. Since at this special point the field H in Eq. (2) cannot be distinguished from h, we conclude that $y_H=y_t=\frac{15}{8}$ and $y=1$ at this f.p., as first conjectured in Ref. 7. This implies that for $J=2K_c$ the scaling and fractal dimensions of the clusters are completely different. This is what has been called Ising droplets behavior in Ref. 7. The crossover between Ising clusters and Ising droplets is described by the relevant exponent y_J of the f.p. at $J=2K_c$ on the line. Comparison with the approximate estimates of Ref. 7 ($y_J\simeq 0.5$), and inspection of the Kac table with $c=\frac{1}{2}$ shows that the most plausible candidate is $y_j=\frac{13}{24}=0.054...$, corresponding to the nonunitary choice of $y_J=2-2\Delta_{3,2}$.

For $J<2K_c$ we have a third regime of critical behavior, in which percolative cluster properties are not anymore becoming critical and $y_H<0$, as we should expect for critical points of Ising type, since only two relevant directions are allowed in that case, and these are already given by the K and h axes.

Our results imply that the full set of exponents describing percolation of Ising clusters at Onsager's critical point cannot be expressed just in terms of known, magnetic Ising indices. Other proposals exist in the literature, which connect exponents like γ, or $\bar{d}$, to magnetic Ising exponents. For example, a recent approach[9] led to $\gamma=\gamma_{\rm Ising}+\beta_{\rm Ising}=1.875$, for the percolative γ exponent. We notice that, if the estimate of Ref. 25 is trusted, this prediction seems possibly to be not an exact one. Most important, however, this result is completely unacceptable in the light of our conformal invariance considerations. Indeed $\gamma=\frac{15}{8}$ implies $y_H=\frac{31}{16}$, a value which cannot follow from the Kac table, with $c=\frac{1}{2}$, for any choice of Δ.

Another conjecture about $\bar{d}$ suggests $\bar{d}=d-\beta_{\rm Ising}/\nu_{\rm Ising}=1.875$.[11] This seems, however, highly implausible. Indeed we should then expect $y_H=\bar{d}=1.875$. This is clearly inconsistent with our numerical results. Moreover, we then would get $\gamma=1.750$, which disagrees with Ref. 25. We saw above that $\bar{d}=\frac{15}{8}$ applies to Ising droplets only.

In summary, in this Letter we have interpreted Onsager's critical point of the $d=2$ Ising model as a tricritical point for correlated percolation, with tricritical $q=1$ Potts exponents. This result follows from conformal invariance arguments and is in remarkable agreement with numerical results. Alternative proposals in the literature are ruled out on the basis of clear numerical evidence and/or incompatibility with conformal invariance.

We mention that further evidence of the correctness of our predictions comes from an analysis of the surface percolative magnetic exponent y_H' of Ising clusters. Numerical results ($y_H'=0.84\pm 0.01$) clearly support the value $y_H'=1-\Delta_{3,3}=0.833...$, which can also be predicted on the basis of conformal invariance.[27]

Conformal invariance also allows to conjecture the exact value of the exponent y_J controlling the crossover between Ising droplets and Ising clusters scaling behaviors.

A.L.S. acknowledges partial support from NATO Grant No. 86/0782 and from the University of Maryland Center for Theoretical Physics and Institute for Physical Science and Technology. He also thanks Professor M. E. Fisher for hospitality and stimulating discussions, and Professor T. L. Einstein for a critical reading of the manuscript.

[1]M. E. Fisher, Physics (N.Y.) **3**, 255 (1967).
[2]C. Domb, J. Phys. A **9**, L141 (1976).
[3]C. Domb, T. Schneider, and E. Stoll, J. Phys. A **8**, L90 (1975).
[4]K. Binder, Ann. Phys. (N.Y.) **98**, 390 (1976).
[5]A. Coniglio, C. Nappi, F. Peruggi, and L. Russo, J. Phys. A **10**, 205 (1977).
[6]H. Muller-Krumbhaar, Phys. Lett. **48A**, 459 (1974).
[7]A. Coniglio and W. Klein, J. Phys. A **13**, 2775 (1980).
[8]D. W. Herrman and D. Stauffer, Z. Phys. B **44**, 339 (1981).
[9]A. D. Bruce and D. J. Wallace, J. Phys. A **16**, 1721 (1983).
[10]T. Temesvari and L. Herenyi, J. Phys. A **16**, L575 (1983).
[11]M. Suzuki, Prog. Theor. Phys. **69**, 65 (1983).
[12]J. L. Cambier and M. Nauenberg, Phys. Rev. B **34**, 8071 (1986).
[13]J. W. Essam, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1973), Vol. 2, p. 197.
[14]P. M. Lam, Koln University report, 1988 (to be published).
[15]B. Derrida and H. J. Herrmann, J. Phys. **44**, 1365 (1983).
[16]B. Nienhuis, in *Phase Transition and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, 1987) Vol. 11, p. 1. See also references therein.
[17]J. L. Cardy, in Ref. 16, p. 55.
[18]K. K. Murata, J. Phys. A **12**, 81 (1979).
[19]H. W. J. Blote, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
[20]Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B240**, 312 (1984); H. Saleur, J. Phys. A **20**, 455 (1987).
[21]A. W. W. Ludwig, Nucl. Phys. **B285**, 97 (1987); A. W. W. Ludwig and J. L. Cardy, Nucl. Phys. **B285**, 687 (1987).
[22]See, e.g., N. Berker and M. Wortis, Phys. Rev. B **14**, 4946 (1976).
[23]H. E. Stanley, J. Phys. A **10**, L211 (1977).
[24]That the exponents of Ising clusters could be given by the Potts tricritical ones was first conjectured in Ref. 10 as an extrapolation of results on Cayley trees.
[25]M. F. Sykes and D. S. Gaunt, J. Phys. A **9**, 2131 (1976).
[26]Runs were of up to 1.25×10^6 Monte Carlo steps per site for $L=36$. Errors were estimated from fluctuations of independent runs. Details will be published elsewhere.
[27]C. Vanderzande and A. L. Stella (to be published).