# Correlations and Fluctuations of Coherent Wave Transmission through Disordered Media

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The correlation functions of the transmission coefficients for scalar wave propagation through disordered media are calculated by use of both diagrammatic techniques and numerical simulations. The calculation is valid in the diffusive regime: multiple elastic scattering with negligible absorption or inelastic scattering and a scattering length much longer than the wavelength. In addition to the familiar large local intensity fluctuations we find a novel memory effect and long-range correlations in the transmission coefficients which decay to a *positive* background value. Implications for light-scattering experiments are discussed.

Coherent waves propagating through a disordered medium[1,2] will emerge from that medium with a phase that varies in an effectively random manner along the wave front. If one assumes that the outgoing wave amplitude at a given point (or in a given mode) is a sum of a large number of uncorrelated amplitudes and that the total amplitudes at points separated by more than a few wavelengths are uncorrelated, then it is easy to show that the wave intensity will fluctuate by an amount of the order of its average.[2,3] The most familiar example of this is the speckle pattern created by a laser beam reflected off a rough surface.[2] It has recently been understood, however, that waves propagating through an inhomogeneous medium by multiple elastic scattering will create a fluctuating intensity pattern which is not nearly as "random" as intuition suggests. First, such patterns will show on average a higher intensity for backscattering as a result of constructive interference of time-reversed pairs of scattering sequences,[1,4] an effect that has been observed recently in several experiments.[5,6] Second, although there are large intensity fluctuations observed,[6,7] the fluctuations at different points (or in different modes) must be statistically correlated. The theoretical and experimental evidence for this comes from the study of conductance fluctuations in small metal circuits.[8-14] It is now understood that these fluctuations occur because of the high sensitivity of the complex interference pattern of the (coherent) electron wave function to changes in impurity scattering potentials. It is found that the fluctuations in the total transmission of electrons through the circuit summed over all incident and outgoing channels (which is proportional to the conductance) is always of order unity.[8-12] It is easy to show that such

a result cannot be obtained if the fluctuations in each mode are uncorrelated.[3,11] It should be stressed that the statistical behavior encountered here is only quantum mechanical in that it derives from the wavelike behavior of electrons, and thus analogous statistical correlations should be present in any system in which waves propagate by coherent multiple elastic scattering. In this Letter we explicitly calculate the statistical correlation function in different transmitted modes and uncover both the correlations responsible for the "universal conductance fluctuations"[8-12] and additional novel correlations which will give a measurable effect in standard wave-transmission experiments such as light scattering.

We consider the propagation of a coherent scalar wave through a $d$-dimensional disordered medium of length $L$ (in the $z$ direction) and cross-sectional area $A = W^{d-1}$, where $W$ is the sample width. Initially we will assume that the waves propagate in a waveguide geometry, with perfectly reflecting walls enclosing the waves in the transverse directions; later we shall discuss the implications of our results for an open geometry. The only important scattering mechanism is assumed to be *elastic*, characterized by an elastic scattering length $l = v\tau$, where $v$ is the wave speed and $\tau$ is the time between scattering events. Since we are primarily interested in studying the correlation and fluctuation behavior of *coherent* wave propagation through a disordered medium, we choose to leave out both inelastic scattering and absorption because these processes destroy the phase coherence of the propagating wave. We also assume that $kl \gg 1$, a condition which is typically valid in light-scattering experiments, and which insures that the waves are not localized. We also take $L \gg l$; i.e., we concen-

trate on the multiple-scattering regime.

The fundamental quantities that we wish to study are the transmission coefficients $T_{ab}$, defined as the ratio of the intensity of the transmitted beam in mode $b$ of the waveguide, to the incident intensity in mode $a$ (in an open geometry $a$ and $b$ will correspond to angles of incidence and transmission), at a fixed frequency $\omega$. Here $a$ and $b$ are integers labeling the transverse eigenmodes of the waveguide outside the disordered region, which are standing waves with transverse wave vectors $q_a = a\pi/W$. The number of propagating channels $N$ is given by $N \approx k^{d-1}A$ ( corresponding to $q_a \le k$ ). Note that for a given sample, i.e., for a given realization of the random positions of all the scatterers, the transmission coefficients are a very complicated function of the positions of all the scatterers; in fact, recent work suggests that they will be sensitive to very small local changes in the scattering potential, equivalent to the motion of a single scatterer.[13,14] The $T_{ab}$ are simply related to the exact Green's function for this sample, $G_{ab}^{\pm}$ for propagation across the disordered region, starting in mode $a$ and ending in mode $b$, by[15]

$$T_{ab} = v_a v_b G_{ab}^{+}(z=0, z=L) G_{ab}^{-}(z=0, z=L), \qquad (1)$$

where $v_a = vk_a/k$, $v_b = vk_b/k$ are the longitudinal velocities in mode $a$ and $b$.

The average transmission coefficient $\langle T_{ab} \rangle$ can be easily computed via diagram Fig. 1(a), where the angular brackets denote an ensemble average over different random configurations for the scatterers, to yield $\langle T_{ab} \rangle = (v_a v_b / v^2) l/NL$. It is useful to define $g = \sum_{ab} T_{ab}$, since in the electron transport problem it can be shown that the conductance $G = 2(e^2/h)g$.[15] Notice that $\langle T_{ab} \rangle \sim 1/L$, which is characteristic of wave propagation in the multiple-scattering regime in the absence of any absorption. We note that $g \approx Nl/L \gg 1$ in the weak-disorder regime $kl > 1$.

We would like to study now how the various $T_{ab}$'s are correlated with one another when the configurations of the scatterers are changed randomly. This can be achieved by considering the correlation function $C_{aba'b'} = \langle \delta T_{ab} \delta T_{a'b'} \rangle$, where $\delta T_{ab} = T_{ab} - \langle T_{ab} \rangle$. To lowest order in the disorder parameter $1/kl$, we obtain

$$C_{aba'b'} = C_{aba'b'}^{(1)} + C_{aba'b'}^{(2)} + C_{aba'b'}^{(3)},$$

where

$$C_{aba'b'}^{(1)} = D_1 \langle T_{ab} \rangle \langle T_{a'b'} \rangle \delta_{\Delta q_a, \Delta q_b} F_1(\Delta q_a L), \qquad (2a)$$

$$C_{aba'b'}^{(2)} = D_2 g^{-1} \langle T_{ab} \rangle \langle T_{a'b'} \rangle [F_2(\Delta q_a L) + F_2(\Delta q_b L)], \qquad (2b)$$

$$C_{aba'b'}^{(3)} = D_3 g^{-2} \langle T_{ab} \rangle \langle T_{a'b'} \rangle, \qquad (2c)$$

where the $D$'s are constants of order unity (except when $W > L$, in which case $D_3 \approx W/L$), $\Delta q_a \equiv |q_a - q_{a'}|$ (similarly for $\Delta q_b$), $F_1(x)$ is a form-factor function
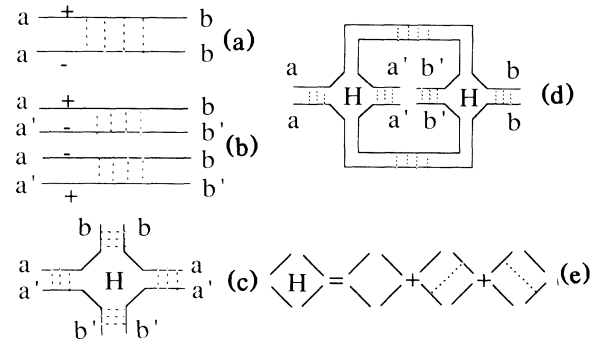


FIG. 1. (a) Feynman diagram for the average intensity $\langle T_{ab} \rangle$. (b) Diagram for $C_{aba'b'}^{(1)}$. (c) Diagram for $C_{aba'b'}^{(2)}$. (d) Diagram for $C_{aba'b'}^{(3)}$. The diamond-shaped vertex in these figures is the Hikami vertex (Ref. 16) shown in (e).

$F_1(x) = x^2/\sinh^2 x$, and

$$F_2(x) = 2x^{-1}[\coth x - x/\sinh^2 x].$$

Note that since $kl \gg 1$ implies $g \gg 1$, the three terms in Eq. (2) are arranged in decreasing order in magnitude.

The above results are derived from the simplest two diffusion diagrams, Fig. 1(b) ($C^{(1)}$ term), by careful use of energy conservation laws (analogous to Ward identities in many-body physics). The techniques used are parallel to those in the calculation of the conductivity tensor $\langle \delta\sigma(r_1, r_2) \delta\sigma(r_3, r_4) \rangle$.[17] Qualitatively, the diagrams shown in Figs. 1(b), 1(c), and 1(d) illustrate the processes which lead to $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$, respectively, although literally speaking the diagrams shown in Figs. 1(c) and 1(d) are divergent.[18] In these diagrams the diffusion ladder represents the process where the waves which originated from different incident or transmitted channels explore the same disorder configuration, and therefore become correlated. We now discuss the significance of these various contributions.

The first term $C^{(1)}$ is only present when $\Delta q_a = \Delta q_b$, and is significant only when $\Delta q_a < 1/L$, because the correlation function, $F_1(x)$, decays exponentially for $x \gg 1$. However, $C^{(1)}$ makes the dominant contribution to $\langle \delta T_{ab}^2 \rangle$, when $\Delta q_a = \Delta q_b = 0$, and its contribution alone gives $\langle \delta T_{ab}^2 \rangle / \langle T_{ab} \rangle^2 = 1$. Thus $C^{(1)}$ represents the familiar large local intensity fluctuations characteristic of speckle patterns.[2,19] The surprising and novel features of the $C^{(1)}$ contribution is that it exhibits what we call a "memory effect"; i.e., it contributes as long as the momentum transfer of the incident beams equals that of the transmitted ones. In the case when a laser beam is transmitted through the sample, this is equivalent to saying that when the incident beam is "tilted" up by a small angle $\Delta\theta$, the transmitted speckle pattern will on average "shift" down by the same angle, provided that $\Delta\theta$ is not too large. This effect is obvious in the context of a single-scatterer problem. That such correlations are still present in the multiple-scattering situation is less ap-

parent intuitively. Moreover, the memory effect decays as a function of $\Delta q\, L \approx \pi L/W$, so that for samples that are wider than they are long it should be possible to measure a sizable contribution from this term.

The second term in Eq. (2), $C^{(2)}$, is smaller than $C^{(1)}$ by a factor $1/g$, but is dominant if *either* the incoming or outgoing channels are nearly the same. Moreover, unlike $C^{(1)}$, it decays very slowly with the momentum difference: $F_2(x) \sim 1/x$. Thus this term exhibits a novel long-range correlation in the transmission fluctuations which is unique to the diffusive regime. This term is dominant in the correlation function between different outgoing channels for a single incident channel. The latter situation has been considered recently by Stephen and Cwilich[20] and our results are in agreement. This term can be understood intuitively as the following: When the two incoming beams are in nearly the same direction, the multiple-scattering paths for the two beams immediately inside the sample are forced to be similar and one would expect a larger correlation in this case.

The third term in Eq. (2), $C^{(3)}$, gives a *uniform positive* correlation of all channels, no matter how widely separated they are in transverse momentum. Because the correlation is uniform, it is quite subtle to detect. In each sample it just causes a shift in the background, i.e., the spatially averaged intensity of each speckle pattern is always a little brighter or darker than the total intensity averaged over many samples, and the crucial point is that these fluctuations are *not* decreased by averaging over a larger and larger spatial region in each pattern. It is this "infinite-range" correlation that leads to the "universal conductance fluctuations." It is readily seen, with our result for $\langle T_{ab} \rangle$, that $C^{(3)} \sim 1/N^4$, and hence that

$$\langle \delta g^2 \rangle = \sum_{a,b,a',b'}^{N} C_{aba'b'} \approx \sum_{a,b,a',b'}^{N} C_{aba'b'}^{(3)} \approx 1.$$

When summed over all channels the contributions of $C^{(1)}$ and $C^{(2)}$ to $\langle \delta g^2 \rangle$ are smaller by a factor $(1/L)^2$ and $(1/L)$, respectively. However, in an experiment where the laser light or microwave is sent into a single channel, and one measures the fluctuations in the total transmitted intensity, the result is dominated by $C^{(2)}$ and is given by $\sum_{aba'b'} C_{aba'b'} \approx 1/NL$, again in agreement with Stephen and Cwilich.[20]

It is straightforward to extend the theory to include correlations between waves at different frequencies, i.e., $\langle \delta T_{ab}(\omega) \delta T_{a'b'}(\omega + \Delta\omega) \rangle$. The finite frequency difference introduces a cutoff in the diffusion pole. The detailed results will be presented elsewhere, but essentially whenever the arguments of $F_1$ and $F_2$ in Eqs. (2a) and (2b) are less than $(\Delta\omega/\omega_c)^{1/2}$, where $\omega_c = D/L^2$, they must be replaced by $(\Delta\omega/\omega_c)^{1/2}$. Equation (2c) is reduced by the same energy correlation factor $F(\Delta\omega/\omega_c)$ found for the universal conductance fluctuations,[11] i.e., $F(x) \approx 1$ for $x \ll 1$ and $F(x) \sim x^{-(4-d)/2}$ for $x \gg 1$.

We have computed the correlation functions numerically by the method of Stone.[21] We found clear support for the three types of behavior predicted by Eq. (2). In Fig. 2(a) we show data for correlation functions of the form $\langle \delta t_{a,b}\, \delta T_{a',b'+\Delta b} \rangle$, first for $a$ and $a'$ very different, and second for $a' = a$. In the first case there is a small positive background correlation (which we interpret as $C^{(3)}$) which increases to a maximum when $\Delta b = b = b'$ ($C^{(2)} + C^{(3)}$); in the second case there is a peak when $\Delta b = 0$ ($C^{(1)} + C^{(2)} + C^{(3)}$), decaying to a background value which is the same as the peak value in the first case. This is exactly the behavior predicted by Eq. (2). These data are for $g \approx 1.2$ as it is difficult to obtain good enough statistics to see the $C^{(2)}$ and $C^{(3)}$ contributions clearly for $g \gg 1$. (We anticipate that this difficulty may persist in light-scattering experiments for measuring the $C^{(2)}$ and $C^{(3)}$ correlations.) This choice of $g$ corresponds to a rather large degree of disorder for which the two-dimensional localization effect[23] is quite observable, and strictly speaking our diagrammatic results derived in the lowest order of $1/g$ do not apply. The fact that the two results agree rather well suggests that higher-order corrections to Eq. (2) are probably small. In Fig. 2(b) we study samples with $g \gg 1$, comparing $\langle \delta T_{a,b}\, \delta T_{a,b+\Delta} \rangle$ with $\langle \delta T_{a,b}\, \delta T_{a+\Delta,b+\Delta} \rangle$ and find rather clear evidence for the memory effect in $C^{(1)}$ (the other terms are too small to be observed with this statistical accuracy). However, we do not find quantitative agreement in that for the latter quantity, Eq. (2a) predicts 0.71 for $\Delta = 1$ whereas the numerical result is only $\approx 0.2$. This discrepancy requires further investigation.
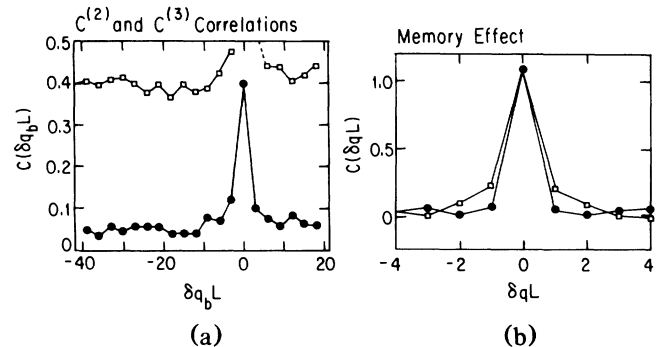


FIG. 2. (a) $C^{(2)}$ and $C^{(3)}$ correlations: $C_{a,b,a',b+\Delta b}$ (circles) and $C_{a,b,a,b+\Delta b}$ (squares) with $a' \gg a$, $\delta q_b L = \pi \Delta b$, for a $20 \times 20$ Anderson model with $g \approx 1.2$ (disorder parameter $w = 4$), averaged over 20000 samples. Circles for $\delta q_b L \gg 1$ conform to the uniform positive background correlation $C^{(3)}$, with error bar $\approx 0.02$. Squares have the same value for $\delta q_b L \gg 1$ as circles at $\delta q_b L = 0$, as predicted by the $C^{(2)}$ term (peak value = 1.71 is omitted). (b) Memory effect: Comparison of $C_{a,b,a',b+\Delta}$ (circles) with $C_{a,b,a+\Delta,b+\Delta}$ (squares) for an $L = 15$, $W = 45$ ensemble of 1500 samples with $g \approx 8.6$ (corresponding to a disorder parameter $w = 2.5$). $\delta q\, L = \Delta \pi L/W$. The value for $\delta q\, L = 1$ is larger in the latter case, as predicted by the $C^{(1)}$ term (error bar is $\approx 0.05$).

Finally, we briefly discuss the implications of our results under various experimental conditions. As our results are derived in a waveguide geometry, a microwave realization is a natural way to verify our detailed predictions, although it may be difficult to send incident waves in channels with $\delta q_a \neq 0$. In the simplest optical experiment, however, one has instead a laser beam of spot size $W$ shining onto an infinitely wide scattering medium of thickness $L$. As long as $W > L$, light diffusion (inside the medium) into areas outside the central region of width $W$ is not severe, and our results for the waveguide geometry should still approximately apply, although the following modifications must be introduced. First, the incoming and the outgoing beam directions are no longer quantized and $T_{ab}$ should be interpreted as the ratio of the integrated outgoing intensity over a solid angle $\Delta \Omega \approx 1/(kW)^{d-1}$ around direction $b$ to that of the incoming beam in direction $a$. Second, since the incoming and outgoing momenta have an uncertainty $\Delta q \approx 1/W$, all the Kronecker delta functions should be understood to have a rounding of $\Delta q$.

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