Space-Time and Topological Orbifolds

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I consider string propagation in a certain complexified version of space-time, in an attempt to gain some intuition about a possible unbroken phase of string theory. Topological σ models (related to Floer and Gromov theory) correspond from this point of view to the consideration of only strings with a self-dual pattern of momenta and windings. I also observe an amusing generalization of the high-energy saddle point studied by Gross and Mende; this generalization involves complex space-time and separate Riemann surface moduli for left-moving and right-moving modes.

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One of the surprising discoveries of recent years is that Yang-Mills instantons play an important role in our understanding of geometry in four dimensions.[1] A closely related development involves the role of σ-model instantons in the understanding of symplectic geometry.[2,3] It has been proposed[4] that quantum field theory (and not just classical instanton theory) is the correct framework for the understanding of these developments. Indeed, renormalizable quantum field theories relevant to the Donaldson, Floer, and Gromov theories can be constructed.[5] In recent works several authors have derived these theories from underlying gauge-invariant actions.[6-9] For my purposes here I will simply work at the Becchi-Rouet-Stora-Tyutin (BRST) gauge-fixed level.

The topological quantum field theories related to Donaldson, Floer, and Gromov theory are not completely unfamiliar. Especially in their gauge-fixed versions, they are closely related to physical constructions. Four is the physical dimension at least macroscopically, and gauge theories, coupled to matter, are the most important quantum field theories in four dimensions, at least at experimentally accessible energies. (In fact, they may well be the only well-defined quantum field theories in four dimensions.) Two-dimensional nonlinear σ models also appear to be rather "physical," in the context of string theory. The most important difference between topological quantum field theories and usual ones is that in the topological theories, the ordinary local excitations are unphysical (or can be gauged away) in the BRST sense. For instance, in topological σ models the graviton vertex operator is a BRST commutator. For this reason, it seems likely that if the topological theories have anything to do with physics, they are related not to the usual phases of physical theories but to a phase in which general covariance and the higher symmetries of string theory are unbroken. The purpose of this paper is to make this idea somewhat more tangible. We will not get very far, but I hope that the discussion will be provocative.

First of all, in topological σ models one must believe that the space-time manifold M is complex (or at least almost complex). This may seem a little bit discouraging at first, but let us proceed anyway. There is some (flimsy) independent evidence[10] that a complexification of space-time is part of what is going on in string theory at a more fundamental level. So we will suppose that the space-time coordinates are d complex variables X^i, $i=1,\dots,d$ (the complex conjugates will be denoted $\bar{X}^{\bar{i}}$). There is no restriction on d, since[5,11,12] there is no critical dimension in topological σ models (the Virasoro anomaly is zero for every matter multiplet and likewise for the gravitational multiplet with its ghosts). The metric will be taken to be flat, $g_{ij}=g_{\bar{i}\bar{j}}=0$, $g_{i\bar{j}}=\mathrm{diag}(-++\cdots+)$. I introduce spin-zero fermions $\psi^i,\psi^{\bar{i}}$ of ghost number one, and fermions $\chi_+^{\bar{i}},\chi_-^{i}$ of ghost number minus one and of spin 1 and -1, respectively. The world-sheet coordinates are denoted as z and $\bar{z}$, respectively. The Lagrangean is

$$\mathcal{L}=\int dz\,d\bar{z}\,(g_{\bar{i}i}\,\partial_z\bar{X}^{\bar{i}}\partial_{\bar{z}}X^i+ig_{\bar{i}i}\chi_+^{\bar{i}}\,\partial_{\bar{z}}\psi^i+ig_{i\bar{i}}\chi_-^i\,\partial_z\psi^{\bar{i}}). \tag{1}$$

This possesses the fermionic symmetry

$$\begin{aligned}&\delta X^i=i\epsilon\psi^i,\quad \delta\bar{X}^{\bar{i}}=i\epsilon\psi^{\bar{i}},\\ &\delta\psi^i=\delta\psi^{\bar{i}}=0,\\ &\delta\chi_+^{\bar{i}}=-\epsilon\,\partial_z\bar{X}^{\bar{i}},\quad \delta\chi_-^i=-\epsilon\,\partial_{\bar{z}}X^i.\end{aligned} \tag{2}$$

(It is actually possible to take separate fermionic parameters for the left- and right-moving modes, but this depends on having a Kahler structure on M, which is a feature that I do not wish to exploit.) I will denote the operator that generates the transformation (2) as Q.

The characteristic feature of (2) is that each field has a Q partner. Related to this, the Hamiltonian is a Q commutator. As a result, the physical states are precisely the ground states. These are tied to the global topology of M. Because M is taken to be contractible, the only physical state is the SL(2)-invariant "vacuum," with vertex operator $V=1$. This might appear unpromising for physical applications. In addition, we must discuss the interpretation of the imaginary part of the space-time coordinates X^i. Perhaps the positions of particles are "really" complex, but the imaginary parts are small in low-energy experiments. The approach in this paper will be, in a sense, to suppress the imaginary parts of the X^i, or at least the zero modes thereof, by the construction of a suitable orbifold. Let G be the group of imaginary shifts of the X^i:

$$X^i \to X^i + 2\pi i k^i. \tag{3}$$

I stress that k^i is real so that only the imaginary part of X^i is being shifted. As we will see, the k^i play roughly the role of momentum vectors. The normalization factor 2π is for convenience. We now consider string propagation on $M'=M/G$. Of course, as a space M' is simply ordinary real Minkowski space. But in string propagation on a quotient M/G, one must introduce the 'twisted sectors," which in the present context are sectors in which the boundary conditions on X^i are

$$X^i(\sigma+\pi) = X^i(\sigma) + 2\pi i k^i, \tag{4}$$

for arbitrary k^i. It is still true that, the Hamiltonian being a Q commutator, only the ground state in any sector of given k^i is physical. Let us denote these states as $|k^i\rangle$, and the corresponding vertex operators as V_{k^i}. Let us study the correlation functions of the V_{k^i}.

In general, in topological σ models, correlation functions are computed by finding suitable instanton solutions. The reason for this is roughly that, according to (1), the instanton action is positive semidefinite and vanishes if and only if the instanton equation

$$\partial X^i/\partial \bar{z} = 0 \tag{5}$$

is obeyed. More fundamentally (as in certain problems in quantum field theories with space-time supersymmetry[13,14]) the fermionic symmetry of (1) can be used[5] to prove that the correlation functions of the BRST-invariant operators are given exactly by a lowest-order instanton calculation.

Let us pick n points $z_1, \ldots, z_n$ on the complex plane, and n shift vectors $k_{(1)}, \ldots, k_{(n)}$, and study the correlation function

$$\langle V_{k_{(1)}}(z_1) \cdots V_{k_{(n)}}(z_n)\rangle. \tag{6}$$

What sort of instanton is relevant? Since $V_{k_{(r)}}$ creates a shift of X by an amount $k_{(r)}$, there must be a cut emanating from each z_r. Across the rth cut, X jumps by $2\pi i k_{(r)}$. Clearly, the holomorphic function with these properties is

$$X^i_{\text{class}} = \sum_{r=1}^{n} k^i_{(r)} \ln(z - z_r). \tag{7}$$

In trying to interpret this instanton physically, there are a few difficulties. First of all, according to (1), the instanton action is zero. Equation (1) is written in a way that makes it obvious that the bosonic part of the action vanishes for instantons. To compare to conventional σ models, note that the bosonic part of (1) can be rewritten as

$$\tfrac{1}{2}\int dz\, d\bar{z}\,(\partial_\alpha \bar{X}^{\bar{i}} \partial^\alpha X^i + \epsilon^{\alpha\beta} \partial_\alpha \bar{X}^{\bar{i}} \partial_\beta X^i). \tag{8}$$

The first term is a conventional σ-model action, but the second term is the usual θ term of the nonlinear σ model at an imaginary value of θ which is just such that the instanton action is zero. This value is the most natural one for topological σ models. Another peculiarity of the present setting compared to usual string theory is that in (6) there is no rationale for integration over the z_r. Perhaps coupling to topological gravity would change this, but this point will not be explored here. Finally, and related to these facts, in (6) there does not seem to be a mass-shell condition on the k_r. Quantization of (1) on M/G gives a physical ground state for every value of k_r. Perhaps one should interpret the mass-shell condition as a property of the broken, low-energy phase of string theory; alternatively, perhaps a mass-shell condition should be obtained by coupling to two-dimensional gravity or shifting θ. It seems pointless to try to "solve" any of these problems at the moment, since the correct assumptions are unclear. Instead I will now try to point out a few additional ingredients of the puzzle.

Gross and Mende[15] calculated the scattering of strings at large angles and very high center-of-mass energies. In many physical theories, such high-energy scattering in the ordinary vacuum is related to properties of a possible high-energy phase—though the nature of the relation varies very much from theory to theory. One of their main conclusions was that high-energy scattering is governed by a particular classical solution (of the world-sheet theory). For scattering of particles of momentum vectors $p_{(r)}$, the relevant classical orbit at tree level is

$$X^i_{\text{GM}} = \frac{i}{2}\sum_r p^i_{(r)} [\ln(z - z_r) + \ln(\bar{z} - \bar{z}_r)]. \tag{9}$$

Clearly, there is a very simple relation between (9) and (7), namely,

$$X^i_{\text{GM}} \sim i\, \text{Re} X^i_{\text{class}}. \tag{10}$$

Thus, the complex instanton X^i_{class} which enters in the study of correlation functions on $M'=M/G$ is closely related to the classical solution that is relevant in high-energy scattering in the ordinary phase of strings. However, (10) cannot be taken too seriously at the present time. The differences between the left- and right-hand sides are numerous; in Ref. 15, the instanton action is nonzero, there is a rationale for extremizing with respect to the z_r, etc. Indeed, X_{GM} has a well-defined relation to a physical calculation, while the introduction of topological orbifolds and the holomorphic instanton X_{class} is just guesswork.

Nevertheless, it is amusing that the imaginary part of X_{class} enters in the study of the high-energy behavior of scattering amplitudes, and it is interesting to ask whether the real part of X_{class} can similarly appear in a plausible calculation of the ordinary, broken phase of string theory. It is easy to think of such a calculation. In principle, there are two ways to obtain a string of very high energy. One can take a string of ordinary microscopic size and accelerate it to very high Lorentz factor; or one can stretch a string to be of cosmic dimensions, obtaining a string which has very high energy because it is very large. Such macroscopic fundamental strings were considered as astronomical objects in another paper[16]; scattering of such macroscopic strings was studied by Polchinski.[17]

Although this is not the usual situation considered for realistic cosmic strings in astronomy, the simplest conceptual framework for thinking about macroscopic fundamental strings is to assume toroidal compactification of spatial dimensions with the cosmic strings as winding states; indeed, this framework was used in Ref. 17. Thus, we consider strings with boundary conditions

$$X^i(\sigma+\pi)=X^i(\sigma)+2\pi a^i, \tag{11}$$

where the a^i are certain shift vectors which are assumed to be of astronomical size. The 2π is for convenience. At this point, if the cosmic string is going to be put on mass shell, one must worry about the time component of the momentum or winding of the string. It is rather unphysical (except in discussion of thermodynamics) to compactify the time direction, and so windings in the time direction are normally impossible and a cosmic string would usually have a very large time component of the center-of-mass momentum. Nevertheless, for conceptual purposes I would like to temporarily imagine scattering of macroscopic strings with windings only and no momenta (and thus, to be on mass shell such strings must have windings in the time direction as well as the spatial directions). Let us consider the scattering of n incident and outgoing cosmic strings with shift vectors $a^i_{(r)}$, $r=1,\ldots,n$. I assume that the corresponding vertex operators W_r are inserted at points z_r on the complex plane. Clearly, the W_r create cuts in the string coordinates X^i with discontinuities $2\pi a^i$. If the a^i are very large, the world-sheet action

$$I=\frac{1}{2\pi}\int dz\,d\bar{z}\,|\partial_a X^i|^2 \tag{12}$$

is very large for any field with these discontinuities, and the cosmic string scattering will be dominated by a classical solution obeying the boundary conditions. Obviously, this classical solution is none other than

$$X_{\text{cosmic}}=\frac{i}{2}\sum_r a^i_{(r)}[\ln(z-z_r)-\ln(\bar{z}-\bar{z}_r)], \tag{13}$$

and this is related to our friend X_{class} by

$$X_{\text{cosmic}}\sim \operatorname{Im}X_{\text{class}}. \tag{14}$$

Thus, in a sense, the imaginary part of X_{class} also has a reasonable rationale in the ordinary phase of string theory. However, there are again many differences in interpretation between (13) and (7). Most notably, in (13) the action (12) is nonzero and very large, and one would want to choose the z_r to minimize it.

In the ordinary phase of string theory, the scattering of a system of strings whose energies are very large in part because of windings and in part because of momenta would be described by a classical solution which is simply the sum of (9) and (13), namely,

$$X^i_{\text{tot}}=\frac{i}{2}\sum_r[(p^i_{(r)}+a^i_{(r)})\ln(z-z_r)+(p^i_{(r)}-a^i_{(r)})\ln(\bar{z}-\bar{z}_r)]. \tag{15}$$

One way to derive this formula is to note that the vertex operator of a string with momentum p and winding a is

$$W_{p,a}=\exp\{\tfrac{1}{2}[(p+a)\cdot X_L+(p-a)\cdot X_R]\}.$$

The scattering process would be given by $\langle\prod W_{p_r,a_r}\rangle$, and to find the expectation value of $X(z)$ in such a process one computes $\langle X(z)\cdot\prod_r W_{p_r,a_r}\rangle$, which is equal to (15) times $\langle\prod W_{p_r,a_r}\rangle$.

At this stage, a somewhat clearer interpretation of (7) can be given. It corresponds to scattering of a system of strings which has a self-dual system of windings and momenta, $p_{(r)}=a_{(r)}$.

This gives a somewhat different perspective on the possible physical meaning of topological σ models. The Lagrangean (1) for strings propagating on M' corresponds to strings with purely self-dual windings and momenta. A parity inversion of (1) would give a reversed Lagrangean that would lead to anti-self-dual windings and momenta. For physical strings which obey neither a self-duality condition nor an anti-self-duality condition, (1) must be combined with its inverse in a suitable way, which remains to be discovered. Of course, the conventional phase of string theory is one very natural system with both left- and right-moving modes, but there may be some sensible combination of (1) and its parity reversal which is closer to a possible unbroken phase and to the crucial geometrical ideas that string theory is based on.

From this point of view, the fact that the BRST-invariant "physics" extracted from topological σ models is purely topological in character would mean that the purely left-moving modes of a string depend only on the topology and are independent of the detailed background, and likewise for the purely right-moving modes. The background dependence would arise in the choice of a way of combining the left- and right-moving modes. Such a suggestion has been made by Peskin in trying to understand background independence in string field theory.[18]

Finally, it is amusing to reconsider the saddle-point calculation of Ref. 15 in the context of the classical solution (15) that incorporates both momenta and windings. One readily sees that the action of (15) is

$$I(z_r,\bar{z}_r) = -\tfrac{1}{2}\sum_{r<s}(p_{(r)}\cdot p_{(s)}+a_{(r)}\cdot a_{(s)})[\ln(z_r-z_s)+\ln(\bar{z}_r-\bar{z}_s)] - \tfrac{1}{2}\sum_{r<s}(p_{(r)}\cdot a_{(s)}+a_{(r)}\cdot p_{(s)})[\ln(z_r-z_s)-\ln(\bar{z}_r-\bar{z}_s)]. \quad (16)$$

[In computing this, one must include the source term $i\sum_r p_{(r)}\cdot X(z_r)$ associated with the momenta; otherwise the $p_{(r)}\cdot p_{(s)}$ term will come out with the wrong sign.] Now, if the $a_{(r)}$ are all zero, then (15) reduces to the saddle point (9) studied by Gross and Mende. A peculiar fact which has not been fully elucidated is that the saddle point is purely imaginary. (It has been suggested[15,19] that a factor of i can be rotated away by a Wick rotation of world-sheet time, leading to an interpretation of X_{GM} as an orbit in *real* Minkowski space.) Even without understanding the significance, if any, of that fact, it is clear that imaginary X means that the action I is real. Likewise, if the $p_{(r)}$ are all zero, then (15) reduces to the real cosmic orbit (13), and for such a real orbit the action is real. But if the $a_{(r)}$ and $p_{(r)}$ are all nonzero, then I is in general complex. This has a dramatic consequence for the saddle-point problem. The saddle-point equations are

$$\partial I/\partial z_r=0, \quad \partial I/\partial \bar{z}_r=0. \quad (17)$$

If I is real, then the second equation is a consequence of the first provided

$$\bar{z}_r=(z_r)^* \quad (18)$$

(that is, $\bar{z}_r$ is the complex conjugate of z_r), which indeed is usually taken as the definition of $\bar{z}_r$ in string theory. But if I is complex, then (17) and (18) would be an overdetermined system; to obey both equations in (17), one must abandon (18) and treat z_r and $\bar{z}_r$ as independent complex variables. Thus a physical process involving the scattering of very-high-energy strings, some of them with high energies because they have been accelerated to high energies and some of them with high energies because they are big, would be governed by a saddle point (generalizing that of Ref. 15) in which z_r and $\bar{z}_r$ would have completely independent complex values. [This does not depend on taking the "big" strings to be winding modes. One likewise is led to complex X^i and the need to abandon (18) if one tries to give a saddle-point description of the scattering of strings that are high up on the leading Regge trajectory, and thus "big."] Abandoning (18) means more or less that the Riemann surface moduli seen by the left-moving modes are independent of the moduli seen by the right-moving modes. Perhaps that is an appropriate state of affairs if at some level space-time is complex and the left- and right-moving contributions to the center of mass of a string are more nearly independent than they are in the usual low-energy phase of string theory.

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