Information-Theoretic Bell Inequalities

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We formulate information-theoretic Bell inequalities, which apply to any pair of widely separated physical systems. If local realism holds, the two systems must carry information consistent with the inequalities. Two spin-s particles in a state of zero total spin violate these information Bell inequalities.

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Local realism is a world view which holds that physical systems have local objective properties, independent of observation. It implies constraints on the statistics of two widely separated systems. These constraints, known as Bell inequalities,[1,2] can be violated by quantum mechanics. The Clauser-Horne-Shimony-Holt[3] (CHSH) Bell inequality applies to a pair of two-state systems and constrains the value of a linear combination of four correlation functions between the two systems. Quantum mechanics violates the CHSH inequality; the violation has been confirmed experimentally.[4]

In this Letter we formulate information-theoretic Bell inequalities. These information Bell inequalities apply to any pair of widely separated systems—not just two-state systems. They are written in terms of the mean information obtained in several measurements on the two systems. Our information Bell inequalities have an appealing interpretation: If local realism holds, the two systems must carry information consistent with the inequalities. The quantum statistics of a pair of spin-s systems in a state of zero total spin violate these information Bell inequalities for arbitrary values of s.

We begin by reviewing pertinent elements of information theory.[5] Consider two measurable quantities (observables) A and B, and label the (discrete) possible values of A and B by a and b. (Throughout we denote an observable by a capital letter and label its possible values by the corresponding lower-case letter.) On the basis of one's knowledge about A and B, one assigns a joint probability $p(a,b)$ for values a and b. Bayes's theorem,

$$p(a,b)=p(a\,|\,b)p(b)=p(b\,|\,a)p(a),$$

relates the joint probability to conditional probabilities.

The information obtained when one discovers values a and b for A and B is $I(a,b)\equiv-\log p(a,b)$. The base of the logarithm determines the units of the information (base 2 for bits, base e for nats). In the same way $I(b)\equiv-\log p(b)$ is the information obtained when one discovers value b for B, and $I(a\,|\,b)\equiv-\log p(a\,|\,b)$ is the further information obtained when one discovers value a for A, if one already knows the value b of B. Written in terms of information, Bayes's theorem becomes $I(a,b)=I(a\,|\,b)+I(b)=I(b\,|\,a)+I(a)$.

Consider now the mean information obtained when one finds values for A and B,

$$H(A,B)=\textstyle\sum_{a,b}p(a,b)I(a,b). \tag{1}$$

This mean information is the entropy of the probability $p(a,b)$; it can be thought of as the total information carried by A and B, defined relative to the knowledge about A and B incorporated in $p(a,b)$. In the same way $H(B)=\sum_b p(b)I(b)$ is the information carried by B, and $H(A\,|\,b)=\sum_a p(a\,|\,b)I(a\,|\,b)$ is the information carried by A, given the value b of B. Averaging $H(A\,|\,b)$ over B gives the conditional information carried by A,

$$\begin{aligned}H(A\,|\,B)&=\textstyle\sum_b p(b)H(A\,|\,b)\\&=\textstyle\sum_{a,b}p(a,b)I(a\,|\,b).\end{aligned} \tag{2}$$

An immediate consequence of Bayes's theorem is the relation

$$H(A,B)=H(A\,|\,B)+H(B)=H(B\,|\,A)+H(A). \tag{3}$$

We require one more ingredient, the mutual information

$$I(a;b)\equiv I(a)-I(a\,|\,b)=I(b)-I(b\,|\,a)=I(b;a),$$

which can be either positive or negative, but whose mean,

$$\begin{aligned}H(A;B)&=\sum_{a,b}p(a,b)I(a;b)\\&=\sum_b p(b)\left[\sum_a p(a\,|\,b)\log\frac{p(a\,|\,b)}{p(a)}\right]\geq0,\end{aligned} \tag{4}$$

is nonnegative.[5] The mean mutual information,

$$\begin{aligned}H(A;B)&=H(A)-H(A\,|\,B)\\&=H(B)-H(B\,|\,A)=H(B;A),\end{aligned} \tag{5}$$

is the information carried in common by A and B.

We need from information theory two inequalities:

$$H(A\,|\,B) \le H(A) \le H(A,B). \quad (6)$$

The left-hand inequality means that removing a condition never decreases the information carried by a quantity. The right-hand inequality means that two quantities never carry less information than either quantity carries separately.

We now turn to the simplest information Bell inequality. Consider two widely separated systems, $\mathcal{A}$ and $\mathcal{B}$, and four measurable quantities—A and A' associated with $\mathcal{A}$, and B and B' associated with $\mathcal{B}$—whose values are denoted by a, a', b, and b'. In a quantum-mechanical description the two observables associated with each system would not commute and hence could not be determined simultaneously. Thus we have in mind a series of experimental runs, in each of which one measures only two quantities, one from each system (as in a test[4] of the CHSH inequality[3]).

Suppose the four quantities are objective, i.e., in each run of the experiment, all four have definite values, independent of observation. These values are unknown, and only two values—one from each system—are determined in each run. Nonetheless, what is known is described by a joint probability $p(a,b',a',b)$, from which follow a pair probability for each measurable pair of quantities—e.g., $p(a,b)=\sum_{a',b'}p(a,b',a',b)$. Since the two systems are widely separated, a measurement on one should not disturb the other. This no-disturbance assumption, based on locality, means that the statistics of runs that measure a particular pair of quantities are given by the appropriate pair probability. More precisely, for runs that measure A and B, the probability of a is $p(a)=\sum_b p(a,b)$, and the probability of b, given a, is $p(b\,|\,a)=p(a,b)/p(a)$. How would a disturbance manifest itself? The conditional statistics of B, given a, would not be those predicted by $p(b\,|\,a)$. Objectivity and locality—a combination called local realism—are used to establish the existence and relevance of the joint probability $p(a,b',a',b)$.

An obvious generalization of the right-hand equality (6) yields

$$H(A,B) \le H(A,B',A',B) = H(A\,|\,B',A',B) + H(B'\,|\,A',B) + H(A'\,|\,B) + H(B), \quad (7)$$

where Eq. (3) is used to expand the right-hand side. A slight generalization of the left-hand inequality (6) can be used to eliminate conditions—i.e., $H(A\,|\,B',A',B) \le H(A\,|\,B')$ and $H(B'\,|\,A',B) \le H(B'\,|\,A')$. Subtracting $H(B)$ from both sides of Eq. (7) leaves the information Bell inequality

$$H(A\,|\,B) \le H(A\,|\,B') + H(B'\,|\,A') + H(A'\,|\,B), \quad (8)$$

which involves pair probabilities that are defined in quantum mechanics and that can be determined from the statistics of the four types of experimental runs. The Bell inequality (8) applies to any four quantities whose statistics can be derived from a joint probability. Its content lies in Eq. (7): Four objective quantities cannot carry less information than any two of them.

To show that quantum statistics violate inequality (8), consider the spin-s generalization[6] of Bohm's version[7] of the Einstein-Podolsky-Rosen paradox.[8] Two counter-propagating spin-s particles, $\mathcal{A}$ and $\mathcal{B}$, having spins $\mathbf{S}_{\mathcal{A}}$ and $\mathbf{S}_{\mathcal{B}}$ (in units of $\hbar$), are emitted by the decay of a zero-angular-momentum particle and thus have zero total spin. Each particle is sent through a Stern-Gerlach apparatus, which measures a component of the particle's spin along one of two possible directions. For particle $\mathcal{A}$ the two observables are $A=\mathbf{S}_{\mathcal{A}}\cdot\mathbf{a}$ and $A'=\mathbf{S}_{\mathcal{A}}\cdot\mathbf{a}'$, where unit vectors $\mathbf{a}$ and $\mathbf{a}'$ specify orientations of the Stern-Gerlach apparatus. The $2s+1$ possible values of A and A', labeled above by a and a', are denoted here by quantum number $m=-s,-s+1,\ldots,s-1,s$. Eigenstates of spin component $\mathbf{S}_{\mathcal{A}}\cdot\mathbf{e}$, where $\mathbf{e}$ is a unit vector, are written as $|sm\rangle_{\mathcal{A},\mathbf{e}}$. Similar notation applies to particle $\mathcal{B}$, with the two spin components specified by unit vectors $\mathbf{b}$ and $\mathbf{b}'$.

The quantum statistics are derived from the state of zero total spin[6]

$$|\phi\rangle = (2s+1)^{-1/2}\sum_{m=-s}^{s}(-1)^{s-m}|sm\rangle_{\mathcal{A},\mathbf{e}}\otimes|s\,-m\rangle_{\mathcal{B},\mathbf{e}}, \quad (9)$$

where the quantization axis $\mathbf{e}$ is arbitrary. Quantum mechanics predicts the probability

$$p(a=m_1,b=m_2) = |{}_{\mathcal{A},\mathbf{a}}\langle sm_1|\otimes{}_{\mathcal{B},\mathbf{b}}\langle sm_2|\,|\phi\rangle|^2 = (2s+1)^{-1}|D_{m_1,-m_2}(R_{\mathbf{n}}(\theta))|^2 \quad (10)$$

that $\mathbf{S}_{\mathcal{A}}\cdot\mathbf{a}$ has value m_1 and $\mathbf{S}_{\mathcal{B}}\cdot\mathbf{b}$ has value m_2. This probability depends only on the angle θ between $\mathbf{a}$ and $\mathbf{b}$. In the rightmost form[6] of Eq. (10),

$$D_{m_1 m_2}(R_{\mathbf{n}}(\theta)) \equiv {}_{\mathcal{A},\mathbf{e}}\langle sm_1|e^{-i\theta\mathbf{n}\cdot\mathbf{S}_{\mathcal{A}}}|sm_2\rangle_{\mathcal{A},\mathbf{e}}$$

is the matrix for a rotation $R_{\mathbf{n}}(\theta)$ by angle θ about any unit vector $\mathbf{n}$ orthogonal to any quantization axis $\mathbf{e}$. For $A=\mathbf{S}_{\mathcal{A}}\cdot\mathbf{a}$ and $B=\mathbf{S}_{\mathcal{B}}\cdot\mathbf{b}$, the quantum-mechanical information $H^{\mathrm{QM}}(A\,|\,B)=H^{\mathrm{QM}}(B\,|\,A)\equiv H^{\mathrm{QM}}(\theta)$ takes the form

$$H^{\mathrm{QM}}(\theta) = -\frac{1}{2s+1}\sum_{m_1,m_2}|D_{m_1,-m_2}(R_{\mathbf{n}}(\theta))|^2\log|D_{m_1,-m_2}(R_{\mathbf{n}}(\theta))|^2. \quad (11)$$

Consider now a special case: the vectors **a**, **b**$'$, **a**$'$, and **b** are coplanar, and successive vectors in this list are separated by angle $\theta/3$. The information Bell inequality (8) is violated if the information difference

$$\mathcal{H}^{\mathrm{QM}}(\theta) \equiv 3H^{\mathrm{QM}}(\theta/3) - H^{\mathrm{QM}}(\theta) \qquad (12)$$

becomes negative, the negative value giving the deficit information carried by the two particles, relative to the requirements of local realism for this geometry.

For small angles ($|\theta| \ll s^{-1}$), $H^{\mathrm{QM}}(\theta)$ is proportional to $-\theta^2 \log\theta^2$. The resulting small-angle behavior of $\mathcal{H}^{\mathrm{QM}}(\theta)$ violates the information Bell inequality (8) for all values of s. The violation is a consequence of the tight correlation between the spins: Knowing the value of $\mathbf{S}_{\mathcal{B}}\cdot\mathbf{b}$ tells one so much about $\mathbf{S}_{\mathcal{A}}\cdot\mathbf{a}$ for **a** near **b** that very little information is gained by determining the value of $\mathbf{S}_{\mathcal{A}}\cdot\mathbf{a}$—so little as to violate the requirements of local realism. To demonstrate the violation, we calculate $\mathcal{H}^{\mathrm{QM}}(\theta)$ for various values of s. The results, displayed in Fig. 1, indicate that the maximum information deficit increases with increasing s, although the range of angles over which there is a violation decreases. (For a two-proton atomic-cascade experiment,[4] where the unit vectors specify orientations of polarization analyzers, one can use the $s=\frac{1}{2}$ plot in Fig. 1 by letting the abscissa be twice the angle between the outermost analyzers.)

The usual Bell inequalities are written in terms of correlations between two two-state systems. The CHSH inequality[3] tested in the most recent experiments[4] involves four quantities, two from each two-state system, it constrains the value of a linear combination of the four measurable correlation functions of these quantities, and it follows from the assumption of a joint probability for the four quantities. Its violation by two spin-$\frac{1}{2}$ particles in a spin-singlet state reflects the tight correlation between the spins. The CHSH inequality is thus closely analogous to the information Bell inequality (8). For the spin orientations considered above, however, the CHSH inequality is violated over a larger range of angles than is inequality (8). Thus the Bell inequality (8) does not reveal all quantum behavior that is inconsistent with local realism.

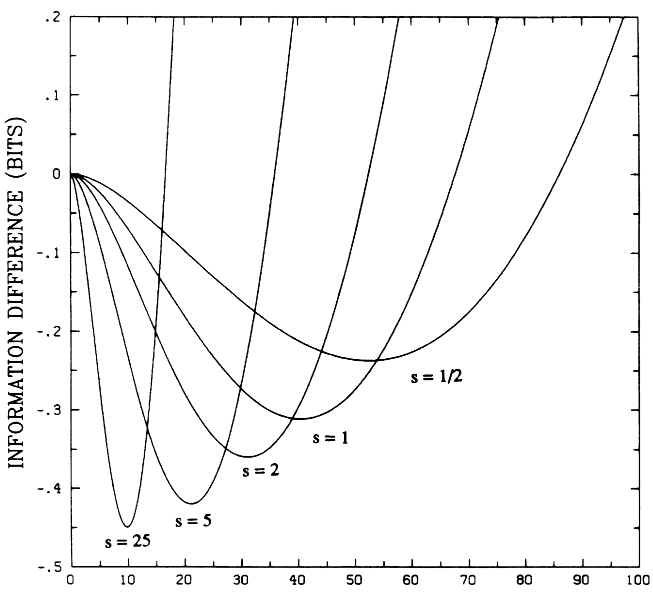

FIG. 1. Information difference $\mathcal{H}^{\mathrm{QM}}(\theta)$ in bits vs angle θ in degrees for $s=\frac{1}{2}$, 1, 2, 5, and 25. The maximum information deficit for $s=\frac{1}{2}$ is -0.2369 bits at 52.31°; for $s=25$, -0.4493 bits at 9.798°.

This realization prompts us to reconsider what it is that Bell inequalities test. A Bell inequality—whether for correlations or for information—is a consequence of our assuming a joint probability for a set of measurable quantities. When quantum mechanics violates a Bell inequality, it means, strictly speaking, only that the quantum statistics cannot be derived from such a joint probability. A Bell inequality is transformed into a test of local realism by the argument that objectivity and realism ensure the existence and relevance of the joint probability. Violation is thus interpreted as a conflict either with objectivity or with locality. (We believe that quantum mechanics conflicts with objectivity because there is no nonlocal disturbance in the sense defined earlier.)

If Bell inequalities arise from a joint probability, why not take a more direct approach? Start with marginal probabilities predicted by quantum mechanics, and ask if they can be derived from higher-order joint probabilities. This approach has been advocated by Garg and Mermin,[9] who formulate it mathematically and investigate it for pairs of spins-s systems for several values of s. The Garg-Mermin approach ferrets out all the consequences of local realism for arbitrary systems, but it is not simple mathematically, nor does it yield clear-cut constraints for experimental test. The CHSH inequality is simple to derive and has been tested, but it does not test all the consequences of local realism, nor is it easy to generalize nontrivially to other than two-state systems. Thus we see a role for information Bell inequalities: They do not get at all the consequences of local realism, but they are simple to derive and applicable to arbitrary systems; as such, they can be a useful tool for the comparison of quantum mechanics against the requirements of local realism.

There is a technique, which we call "chaining," for making more onerous the requirements of local realism. Chained correlation Bell inequalities have been considered by Selleri and Tarozzi[2] and by Garuccio and Selleri,[10] but not much discussed. Here we apply chaining to the information Bell inequality (8).

Consider as before two widely separated systems, $\mathcal{A}$ and $\mathcal{B}$, and $N=2Q$ measurable quantities—$A_1,\ldots,A_Q$ associated with $\mathcal{A}$ and $B_1,\ldots,B_Q$ associated with $\mathcal{B}$—interleaved in a sequence $A_1,B_Q,A_2,B_{Q-1},\ldots,A_{Q-1},$

B_2, A_Q, B_1. Objectivity and locality justify a joint probability for the N quantities. A trivial extension of the reasoning that leads to inequality (8) establishes the chained information Bell inequality

$$H(A_1|B_1) \leq H(A_1|B_Q) + H(B_Q|A_2) + \cdots + H(B_2|A_Q) + H(A_Q|B_1), \tag{13}$$

called "chained" because it can be obtained by repeated application of the $N=4$ inequality (8). Local realism becomes more burdensome as N increases because the increasing number of objective quantities requires the system to carry more information.

To investigate violation by quantum mechanics, return to the two spin-s particles considered above. The N measurable quantities, $A_j = \mathbf{S}_{\mathcal{A}} \cdot \mathbf{a}_j$ and $B_j = \mathbf{S}_{\mathcal{B}} \cdot \mathbf{b}_j$ for $j = 1, \ldots, Q$, are spin components specified by unit vectors $\mathbf{a}_1, \mathbf{b}_Q, \mathbf{a}_2, \ldots, \mathbf{b}_2, \mathbf{a}_Q, \mathbf{b}_1$. If these vectors are coplanar and successive vectors in the list are separated by angle $\theta/(N-1)$, then the chained information Bell inequality is violated when the information difference

$$\mathcal{H}_N^{\rm QM}(\theta) \equiv (N-1) H^{\rm QM}(\theta/(N-1)) - H^{\rm QM}(\theta) \tag{14}$$

is negative. For any θ, when N is sufficiently large, $H^{\rm QM}(\theta/(N-1))$ can be approximated by the small-angle behavior. In the limit $N \to \infty$, $\mathcal{H}_N^{\rm QM}(\theta) \to -H^{\rm QM}(\theta) \leq 0$, which violates inequality (13) for all θ except multiples of π. For large-N chaining, the $N-1$ measurements at the small angle $\theta/(N-1)$ together yield vanishingly small information because of the tight correlation between the spins; this vanishing of information is closely related to the quantum Zeno paradox.[11]

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