Quantum-Classical Correspondence in Many-Dimensional Quantum Chaos

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(Received 5 April 1988)

Quantum-classical correspondence in many-dimensional quantum chaos is investigated by use of a coupled quantum kicked-rotors model. Even when the number of rotors is only two, results obtained are drastically different from those for a single-rotor system; that is, in the semiclassical limit the coupled system restores the essential features of classical chaos under appropriate conditions. In particular, the time-reversal experiment reveals that the classical chaotic mixing is recovered almost entirely; however, the recovered mixing is "conditional" in the sense that there exists a threshold for the recovery.

PACS numbers: 05.45.+b

Since the numerical experiment by Casati *et al.*,[1] it has been recognized that quantum-classical correspondence in chaotic dynamics seems to be quite anomalous.[2-6] The time scale for which the quantum-classical correspondence holds is too short. It is conjectured to be as short as $T_r \propto \log\hbar/\lambda$ (λ, positive Lyapunov exponent).[2,4,5] Beyond this time scale the quantum chaos system becomes quite stable and loses the ability of mixing.[6,7] However, studies of quantum chaos have been so far restricted to few-dimensional systems with at the most three dimensions. There is a possibility that in many-dimensional quantum chaos the quantum-classical correspondence may be recovered far more naturally than in a few-dimensional one. Indeed, recent studies reveal that the few-dimensional quantum chaos under a continuous application of classical noise restores not only the ergodicity[5,7,8] but also the mixing[9] possessed by its classical counterpart. These facts lead us to the idea that coupled quantum chaos systems may cooperatively recover the nature of classical chaos. In the present paper we report the first study of quantum-classical correspondence in many-dimensional quantum chaos by use of a coupled quantum kicked-rotors model. The results obtained are quite drastic even when the number of rotors is only two. We, therefore, confine ourselves to a two-coupled-rotors system hereafter. Studies for more than three coupled rotors will be reported in subsequent papers.[10]

The system we consider is a coupled N-kicked-rotors system described by the Hamiltonian

$$\hat{H}=\hat{T}(\{\hat{p}_i\})+\sum_{n=-\infty}^{+\infty}\hat{V}(\{\hat{\theta}_i\})\delta(t-n), \qquad (1)$$

where $\hat{\theta}_i$ and $\hat{p}_i=-i\hbar\,\partial/\partial\hat{\theta}_i$ are the position and momentum operators and t is time. In the present paper we assume the specific forms of the kinetic and potential energies as $\hat{T}(\{\hat{p}_i\})=\sum_i\hat{p}_i^2/2$ and

$$\hat{V}(\{\hat{\theta}_i\})=\textstyle\sum_i K_i\cos\theta_i+\sum_{ij}\epsilon_{ij}\cos(\hat{\theta}_i-\hat{\theta}_j).$$

If the coupling constant ϵ_{ij} is zero, the dynamics of a single uncoupled rotor is described by the standard mapping.[11] In quantum dynamics a single-step evolution $(t\rightarrow t+1)$ is attained by the operation of the unitary operator $\hat{U}=\hat{U}_1\hat{U}_2$, where $\hat{U}_1(\{\hat{p}_i\})=\exp[-i\hat{T}(\{\hat{p}_i\})/\hbar]$ and $\hat{U}_2(\{\hat{\theta}_i\})=\exp[-i\hat{V}(\{\hat{\theta}_i\})/\hbar]$.

In numerical computation we used the fast-Fourier-transformation (FFT) method to save CPU time: First, apply the operator $\hat{U}_2(\{\hat{\theta}_i\})$ to the wave function of the position representation. Next, transform the wave function into the momentum representation by the FFT, and apply the operator $\hat{U}_1(\{\hat{p}_i\})$, which is diagonal in the momentum representation. Finally, transform the wave function again into the position representation by the FFT. By this method the CPU time is reduced about $\log(N_L/N_L^N)$ (N_L, number of quantum levels for a kicked rotor) times shorter than by the ordinary method.

Let us first examine a slightly specific case to investigate whether the coupled quantum chaos may restore the nature of classical chaos. To this end we set $K_1=K_2$ $(\equiv K)$ above the classical stochastization threshold K_C ~ 0.97. Above K_C the classical uncoupled rotor exhibits a chaotic diffusion across the momentum space.[11] In contrast to this a single quantum rotor is quite stable[5] and does not exhibit diffusion except for in a very initial stage of time evolution.[1-3] In Fig. 1(a) we show typical examples of the time evolution of the moment $M^{(i)}(t)$ $=\langle\psi_0|(\hat{p}_t-\hat{p}_0)^2|\psi_0\rangle$ $(\hat{X}_t\equiv\hat{U}^{-t}\hat{X}\hat{U}^t$; ψ_0, initial wave function). As is expected from the behavior of a single quantum rotor,[1,3] $M^{(i)}$ tends to saturate at a finite level when ϵ is small enough. As ϵ increases, the saturation level grows rapidly, and the transient behavior of $M^{(i)}(t)$ shows a fractional diffusion $M^{(i)}(t)\propto t^\beta$ $(\beta<1)$. With further increase in ϵ, the exponent β reaches 1 and normal diffusion appears! To make a quantitative comparison with the classical result, we depict in Fig. 1(b) the ϵ dependence of the diffusion constant. For $\beta<1$, the time evolution of $M^{(i)}(t)$ cannot be characterized by a single diffusion constant, and so we introduce the moment-dependent diffusion constant $D^{(i)}(M)$ defined by $D^{(i)}(M)=[dM^{(i)}/dt]_{t=t_M}$, where t_M is determined by $M^{(i)}(t_M)=M$. M is varied between two values M_1 $\ll M_2$ $[\ll(\Delta P)^2$, where ΔP is the size of momentum

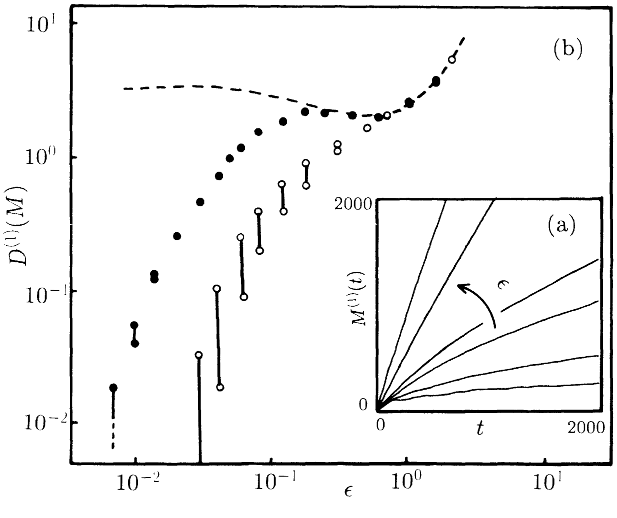

FIG. 1. (a) Variation of the functional form of the moment $M^{(1)}(t)$ as ϵ is increased. $\hbar/2\pi=\frac{41}{512}$. (b) ϵ dependence of $D^{(1)}(M)$. Here M is varied between $M_1=300$ and $M_2=1000$. Filled circles, $\hbar/2\pi=\frac{41}{1024}$. Open circles, $\hbar/2\pi=\frac{41}{512}$. Dashed line, classical diffusion constant.

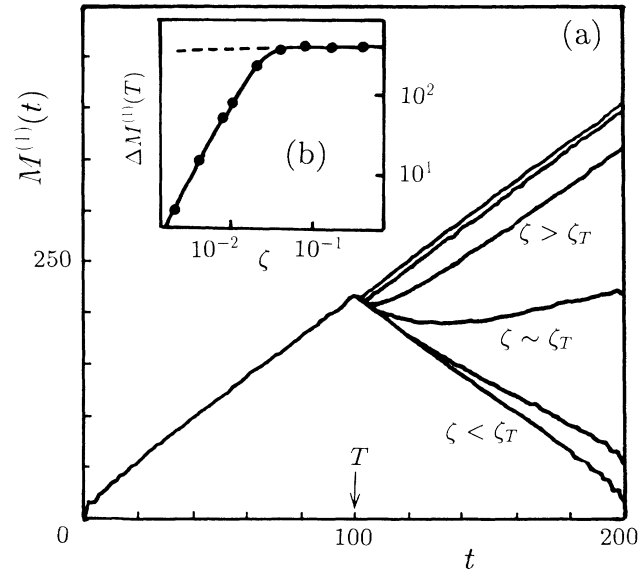

FIG. 2. Time-reversal experiment. (a) Time evolution of $M(t)$ for various ζ before $(t<T)$ and after $(t>T)$ the time-reversal operation. (b) ζ dependence of the time irreversibility. Classical result is indicated by the dashed line. Here $K_1=K_2=3.0$, $\epsilon=0.4$, and $\hbar/2\pi=\frac{21}{512}$.

space]. For small ϵ, $D^{(i)}(M)$ is quite small and spreads between $D^{(i)}(M_1)$ and $D^{(i)}(M_2)$ because of the fractional diffusion. With an increase in ϵ $D^{(i)}(M)$ increases and its spread decreases quickly. Finally the normal diffusion characterized by a single diffusion constant appears, and above a certain threshold ϵ_C the diffusion constant becomes in entire agreement with the classical diffusion constant $D_{\rm cl}$. We note here that the $M^{(i)}(t)$ for the uncoupled rotor cannot reach even to the lower level M_2 ($M_2=300$) for $t\to\infty$. Thus it is quite apparent that a drastic recovery of a classical diffusion occurs in the coupled rotors. Since the momentum space is bounded, the diffusive motion begins to saturate as $M^{(i)}(t)$ reaches $\approx(\Delta P)^2/40$. We confirmed the persistence of diffusive motion by increasing the size of momentum space ΔP up to $41\times2\pi$. Furthermore, we computed $M^{(i)}(t)$ for the classical system under the same boundary condition as the quantum system and compared it with the quantum $M^{(i)}(t)$. For $\epsilon\gg\epsilon_C$ the results agree quite well even at the saturation level. In this way we confirmed the recovery of classical behavior.

The "classicalization" threshold ϵ_C is quite small in the semiclassical limit. Using the mixing characteristics for a single rotor under an application of classical noise,[9] we can develop a simple self-consistent theory to determine ϵ_C. In the present paper we do not go into the detail of the theory and only refer to the result. ϵ_C is evaluated as $\epsilon_C\sim(K/2D_{\rm cl}T_r)^{1/2}\hbar$.[10]

If the motion of one of the two rotors becomes classically chaotic, it perturbs the other rotor as if it were a classical noise source, which makes the motion of the other rotor classically chaotic.[9] Hence there is a positive feedback inducing the classical chaotic motion. Such a cooperative interaction restores the classical time-dependent behavior if the coupling is strong enough.

Existence of diffusion does not necessarily mean recovery of classical mixing. To test the presence of mixing the time-reversal experiment is most useful[9]: Evolve the system for a finite period T and apply a perturbation at $t=T$. Next evolve the system back by the time-reversed evolution rule for T. The difference $\Delta M^{(i)}(T)=M^{(i)}(2T)-M^{(i)}(0)$ characterizes the irreversibility of the system. We assume that the perturbation is described by the Hamiltonian $\zeta(\hat{p}_1+\hat{p}_2)\delta(t-T)$, which makes the positions of the rotors shift by ζ. A single quantum rotor is quite stable[5] and does not exhibit irreversibility except in a very initial stage of time evolution, i.e., $T\lesssim T_r$.[10] However, as shown in Fig. 2(a), a drastic change occurs for the coupled quantum rotors system. If ζ is not too small, the time-reversed process $(t>T)$ loses the memory necessary to return to the initial state immediately after $t=T$, and the time reversibility is entirely violated. This implies a recovery of classical chaotic mixing. However, unlike the classical chaotic mixing, there is a threshold ζ_T for the perturbation strength ζ. As shown in Fig. 2(b), below ζ_T the irreversibility $\Delta M^{(i)}(T)$ is dramatically reduced from the classical value. Classically interpreted, the presence of the threshold means that the loss of memory due to the orbital instability occurs only when the initial distance between the two nearby orbits is more than a critical length determined by ζ_T. In this sense the recovery of mixing is "conditional." The presence of such a threshold is an essential feature of the chaotic mixing in quantum sys-

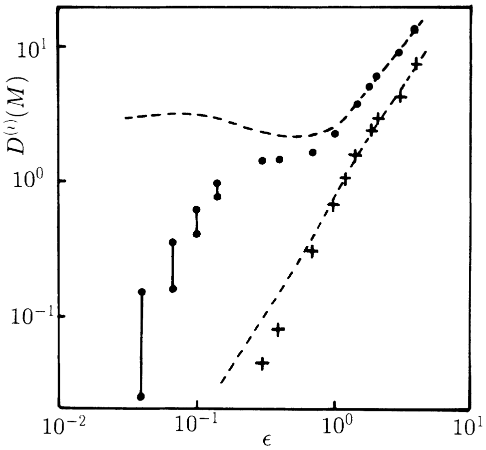

FIG. 3. ϵ dependence of $D^{(1)}(M)$ (circles) and $D^{(2)}(M)$ (crosses) compared with classical results (dashed lines). Here $K_1=3.0$ and $K_2=0.8$. M is varied from 300 to 700 for rotor 1 and from 50 to 200 for rotor 2, respectively. Here $\hbar/2\pi=\frac{41}{1024}$.

tems. We numerically confirmed that in the regime where the quantum-classical correspondence is recovered ζ_T is proportional to $\hbar$ and the ratio $\zeta_T/\hbar$ seems to be insensitive to ϵ and ΔP.

A fundamental question is whether many-dimensional chaos in general restores the nature of classical chaos. In this respect, the example we discussed so far is rather specific because in the classical limit the constituent rotors are already in chaotic states before the interaction is introduced. To answer the above question we consider a more general case, i.e., $K_1>K_C$ and $K_2<K_C$. If $\epsilon=\epsilon_{12}=0$ the Lyapunov spectrum of the classical counterpart has only one positive exponent, but as ϵ increases the second exponent increases monotonically and chaotic motion becomes two dimensional. Simultaneously the second rotor exhibits a significant diffusive motion. In Fig. 3 we show examples of $D^{(i=1,2)}(M)$ as functions of ϵ. As ϵ exceeds a certain threshold ϵ_C, the diffusion constants agree with the classical diffusion constants. The threshold ϵ_C is much larger and less sensitive to $\hbar$ than the case of $K_1>K_C$ and $K_2>K_C$. The time-reversal test reveals that the conditional mixing is recovered also in this case. This experiment strongly suggests that if the corresponding classical system has two sufficiently large Lyapunov exponents the nature of classical chaos is recovered in its quantal counterpart.

A plausible explanation may be as follows: Let us first consider the single-rotor system.[7] If the system is classically chaotic, the support of the wave function is expanded and folded with time along a one-dimensional chaotic manifold in the two-dimensional phase space. Thus the distribution of the wave function in the phase space soon becomes dense enough to induce self-interference everywhere in the phase space. This prevents further development of the chaotic structure.[7] In contrast to this, for a system of two coupled rotors the dimension of phase space is four, and the support of the wave function is expanded and folded like a two-dimensional sheet if two Lyapunov exponents are positive. Since the two-dimensional sheet in the four-dimensional space is relatively more sparse than the one-dimensional curve in the two-dimensional space, the self-interference of the wave function occurs much less efficiently than in the case of the single rotor, and is unable to prevent the chaotic evolution. The above explanation is, however, only a conjecture and needs to be examined more severely. This will be done in forthcoming publications.

In conclusion we have shown that the nature of many-dimensional quantum chaos is drastically different from that of few-dimensional quantum chaos. In particular the chaotic mixing is restored at least conditionally under appropriate conditions.

This work was supported through a Grant-in-Aid for Scientific Research provided by the Ministry of Education, Science and Culture, Japan.

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