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Potential for Mixing in Quantum Chaos

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A study of time reversibility reveals that a quantum chaotic system manifests the same potential for mixing as its classical counterpart under the influence of externally applied noise. The quantum-chaotic system, however, reveals its own nature below a quite small "classicalization" threshold of noise intensity: In the case of the kicked rotor there exists a quite wide regime for which the mixing property is understood neither from classical dynamics nor from quantum perturbation theory. The relationship between such a curious behavior and the pure quantum evolution without the noise is elucidated.

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There seems to be a serious discrepancy between classical chaos and its quantum counterpart (called "quantum chaos" hereafter). Indeed, the dynamical behavior of quantum chaos cannot simulate its classical counterpart except for a quite limited initial stage of time evolution.[1–3] In particular, a crucial fact is that quantum chaos is quite stable[4] and does not exhibit mixing. This is because quantum motion is a quasiperiodic oscillation determined by eigenfrequencies that form a pure-point-set spectrum.[5,6] However, it has recently been shown that a continuous application of a very small dynamical perturbation, such as random noise[4,7] or the measurement process,[8] modifies severely the nature of quantum chaos and enables it to regain the ergodicity possessed by its classical counterpart. It is, therefore, a quite natural question to inquire whether quantum chaos can regain the mixing of its classical counterpart when it is subjected to the continuous influence of dynamical perturbations.

The aim of the present paper is to demonstrate that quantum chaos under an application of very weak external noise can completely simulate the mixing property of its classical counterpart. There exists, however, a certain "classicalization" threshold of noise level below which quantum chaos reveals its own nature. We investigate this regime in detail for the kicked rotor: The perturbation approach is applicable only at noise levels much lower than the classicalization threshold and there is a quite wide regime where the characteristics of mixing are explained in terms of a transient behavior inherent in the pure quantum evolution process without the noise.

How can we characterize the quantum mixing quantitatively? The decay of a correlation function provides the most definite evidence for the presence of mixing. It is, however, not easy to judge numerically whether the decay of correlation is intrinsic or not. In view of adaptability to numerical simulation time reversibility is easier to compute,[4] and, moreover, it reflects the presence of mixing sensitively, as discussed later. We are interested in the time reversibility of quantum chaos under the continuous application of external noise: Evolve the system forward from an initial state Ψ_i for a finite period T by the normal time-evolution rule with an external noise. Next, evolve the system back to a final state Ψ_f by the time-reversed evolution rule for T. The noise process is not reversed. Then the difference ΔX between the expectation values of an observable $\hat{X}$ in the final and initial states,

$$\Delta X(T,\epsilon)=\langle\langle\Psi_f|\hat{X}|\Psi_f\rangle-\langle\Psi_i|\hat{X}|\Psi_i\rangle\rangle,$$

characterizes the time irreversibility of the system. Here ϵ is the average noise intensity and the boldface angular brackets indicate an averaging over the noise processes.

In classical chaos there exists at least two time scales.

The first time scale T_{C_1} is the one on which the system loses its initial memory. It is roughly estimated to be $T_{C_1} \sim \ln\epsilon/\lambda$ (λ is the largest Lyapunov exponent). The second time scale is the one on which the noise causes the system to diffuse out of the chaotic manifold. It is estimated to be $T_{C_2} \sim \epsilon^{-2}$. Choosing T so that $T_{C_1} \ll T \ll T_{C_2}$ (say, $T \sim \epsilon^{-1}$), we see that $\Delta X(T,\epsilon)$ jumps discontinuously from 0 to some finite value as ϵ is increased from 0. This is a manifestation of the fact that any perturbation approach breaks down at $\epsilon=0$. In contrast to this, the perturbation theory works well for integrable systems and $\Delta X(T,\epsilon)$ has no discontinuity at $\epsilon=0$. $\Delta X(T,\epsilon)$, therefore, sensitively reflects the presence of the mixing in classical systems.

In the present paper we consider the well-known kicked rotor as a sample system. The Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2} + K\cos\hat{\theta} \sum_{n=-\infty}^{+\infty} \delta(t-n-\zeta_n).$$

Here $\hat{p} = -i\hbar\,\partial/\partial\hat{\theta}$ and $\hat{\theta}$ are the momentum and position operators, and ζ_n is the externally applied frequency-modulation noise with the statistical property $\langle\zeta_n\rangle = 0$, $\langle\zeta_n\zeta_{n'}\rangle = \zeta^2\delta_{nn'}$. A single-step evolution ($t=n+\zeta_n \rightarrow t=n+1+\zeta_{n+1}$) is attained by operating with the unitary operators of the pure process

$$\hat{U} = \exp(-i\hat{p}^2/2\hbar)\exp(-iK\cos\hat{\theta}/\hbar)$$

and of the noise process $\hat{P}(n) = \exp(-i\zeta_n\hat{p}^2/2\hbar)$. The reason why we take the kicked rotor as a sample system is that it is globally translationally invariant in momentum space, and the irreversibility

$$\Delta X(T,\epsilon) = \langle\langle\Delta\hat{p}^2\rangle_f - \langle\Delta\hat{p}^2\rangle_i\rangle,$$

where $\Delta\hat{p} \equiv \hat{p} - \langle\hat{p}\rangle_i$ and $\langle\hat{X}\rangle_{i,f} \equiv \langle\Psi_{i,f}|\hat{X}|\Psi_{i,f}\rangle$, is characterized by the diffusion constant $D_Q(\epsilon) = \lim_{T\to\infty}\Delta X(T,\epsilon)/2T$, which is independent of the reversal time T. The $D_Q(\epsilon)$ agrees with the diffusion constant in the ordinary sense which is defined only for the forward process, i.e., $\lim_{T\to\infty}\langle\langle\Delta\hat{p}^2\rangle_f\rangle/T$. This has been verified through extensive numerical simulations. To simplify the problem we assume that the initial-state momentum p_0 is large enough that $\hat{P}(n)$ can be replaced by $\exp(-i\epsilon_n\hat{p}/\hbar)$, where $\epsilon_n = 2\zeta_n p_0$ ($\langle\epsilon_n\epsilon_{n'}\rangle = \delta_{nn'}\epsilon^2$). Then the classical counterpart of the kicked rotor is described by the noise-driven standard map

$$(\theta_{n+1}, p_{n+1}) = (\theta_n + p_{n+1} + \epsilon_n, p_n + K\sin\theta_n).$$

The classical dynamics of the standard map undergoes a transition to global chaos as K exceeds the critical value $K_C = 0.97\ldots$, which leads to diffusion across momentum space with $(p_t - p_0)^2 = D_{\mathrm{Cl}}t$.[1] In the quantum motion, however, such a diffusion saturates at $(\delta p)^2 \sim D_{\mathrm{Cl}}/\hbar^2$,[1,9] because of Anderson localization of the quasienergy eigenstates of the operator $\hat{U}$.[6] Here δp is the localization length of the eigenstates in momentum space.

At K well below K_C, the system is almost integrable and the classical-quantum correspondence holds quite well. Indeed, the numerically computed quantum irreversibility $D_Q(\epsilon)$ agrees quite well with the classical one $D_{\mathrm{Cl}}(\epsilon)$ for small $\hbar$. Furthermore, perturbation theory works well until $\epsilon \simeq 1$ in both the quantum and classical cases. Hence $D_{\mathrm{Cl}}(\epsilon) = D_Q(\epsilon) \sim \epsilon^2$ and there is no discontinuity at $\epsilon=0$.

In the chaotic regime $K > K_C$, $D_{\mathrm{Cl}}(\epsilon)$ jumps discontinuously from 0 to the classical chaotic diffusion rate $D_{\mathrm{Cl}}(\epsilon=+0)$ at $\epsilon=0$. In contrast to this, $D_Q(\epsilon)$ has no discontinuity at $\epsilon=0$. However, as depicted in Fig. 1, $D_Q(\epsilon)$ becomes in complete agreement with $D_{\mathrm{Cl}}(\epsilon)$ when ϵ exceeds a classicalization threshold ϵ_C. Usually the effect of noise is supposed to assist the mixing. However, a quite interesting fact is that $D_{\mathrm{Cl}}(\epsilon)$ decreases considerably from $D_{\mathrm{Cl}}(\epsilon=+0)$ before it is enhanced again by the noise-assisted mixing. The decrease of $D_{\mathrm{Cl}}(\epsilon)$ implies that the noise partly damages the classical horseshoe dynamics which is the essential origin of chaotic mixing.[10] This phenomenon may therefore be called *noise-suppressed mixing*.[11] The fact that above ϵ_C $D_Q(\epsilon)$ reproduces entirely the details of $D_{\mathrm{Cl}}(\epsilon)$ including the noise-suppressed mixing means that noise restores the classical horseshoe dynamics in quantum chaos. Thus quantum chaos has the same potential for mixing as its classical counterpart under the influence of noise. The existence of such mixing characteristics is being verified also in other quantum-chaotic systems.

Below ϵ_C, however, the quantum system reveals its own nature: $D_Q(\epsilon)$ is found to decrease according to a power law $D_Q(\epsilon) \propto \epsilon^\nu$ with an exponent $\nu \simeq 1.0$. Such a behavior cannot be explained at all by the standard perturbation theory. Indeed perturbation theory predicts[7]

$$D_Q(\epsilon) = (\epsilon^2/\hbar^2)\sum_\beta |\langle\alpha|\hat{p}|\beta\rangle|^2(\langle\beta|\Delta\hat{p}^2|\beta\rangle - \langle\alpha|\Delta\hat{p}^2|\alpha\rangle) \sim \epsilon^2\delta p^4/\hbar^2,$$

where $|\alpha\rangle$ and $|\beta\rangle$ are the eigenstates of $\hat{U}$. If $D_Q(\epsilon) \propto \epsilon^\nu$ persists until $\epsilon=0$, then perturbation theory breaks down also in the quantum-chaotic system. This, however, cannot be true considering that all the eigenstates are localized and the quantum spectrum forms a pure point set.[6]

The results of careful numerical simulations at a quite small noise level are summarized in Fig. 2. Evidently $D_Q(\epsilon)$ crosses over at a certain $\epsilon=\epsilon_T$ from $D_Q(\epsilon) \sim \epsilon^\nu$ to the perturbative result $D_Q(\epsilon) \sim \epsilon^2$. Thus there exist three regimes of noise level: classical regime, $\epsilon \gg \epsilon_C$, $D_Q(\epsilon) = D_{\mathrm{Cl}}(\epsilon)$; quantum regime, $\epsilon_T \ll \epsilon \ll \epsilon_C$, $D_Q(\epsilon) \propto \epsilon^\nu (\nu \simeq 1.0)$; and perturbative regime, $\epsilon \ll \epsilon_T$, $D_Q(\epsilon) \propto \epsilon^2$. As shown in Fig. 2 both ϵ_C and the ratio ϵ_T/ϵ_C decrease progressively with decrease in $\hbar$. The perturba-

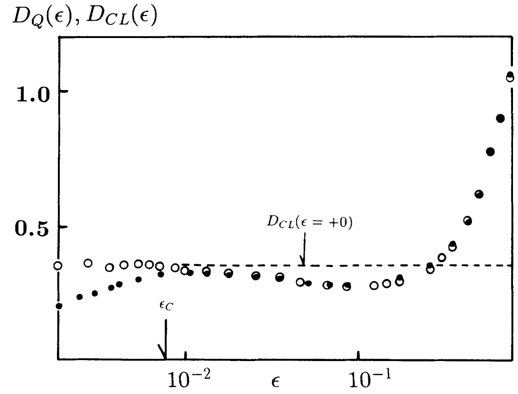

FIG. 1. Classical (open circles) and quantum (filled circles) irreversibilities at relatively large noise intensity ($\hbar/2\pi = \frac{21}{4096}, K=2.0$).

tion theory thus breaks down above ϵ_T, which is much smaller than ϵ_C.

In what follows we discuss the underlying mechanism leading to the characteristics mentioned above. This enables us to determine the threshold values ϵ_C and ϵ_T. Let us introduce the force-force correlation function $g(\epsilon,t) \equiv K^2 \langle \sin\hat{\theta}_t \sin\hat{\theta}_0 \rangle$ and the area defined by $a(\epsilon,t) \equiv \sum_{s=-t}^{t} g(\epsilon,s)$. Here $\hat{X}_t \equiv \hat{V}(t)^{-1}\hat{X}\hat{V}(t)$, and $\hat{V}(t)$ is the evolution operator $\hat{V}(t) \equiv \prod_{i=0}^{t} \hat{U}\hat{P}_i$. Then the Heisenberg equation $\hat{p}_{t+1} - \hat{p}_t = K \sin\hat{\theta}_t$ leads to the relation $D_Q(\epsilon) = a(\epsilon, t=\infty)$. On the other hand, the moment $M_\epsilon(t) \equiv \langle \Delta\hat{p}_t^2 \rangle$, which is expressed in terms of $a(\epsilon,t)$ as $M_\epsilon(t) = \sum_{s=0}^{t} a(\epsilon,s)$, saturates at $(\delta p)^2$ for $\epsilon=0$ beyond a characteristic time T_s (Anderson localization). This means

$$a(\epsilon=0,t) \underset{t \gg T_s}{\longrightarrow} 0$$

for $\epsilon=0$. For finite ϵ, however, the effect of noise modifies significantly the quasienergy of $\hat{V}(t)$ in a random manner, and the correlation function $g(\epsilon,t)$ decays on a certain time scale τ_c. τ_c is estimated in such a way that the accumulated phase in the product of the perturbation operators $\hat{p}\sum_{i=1}^{\tau_c} \epsilon_i/\hbar \sim \delta p \epsilon \sqrt{\tau_c}/\hbar$ amounts to $\simeq 1$. This yields $\tau_c \sim \kappa/\epsilon^2$ ($\kappa \equiv \hbar^2/\delta p^2$). Since $g(\epsilon,t)$ may be replaced by $g(\epsilon=0,t)$ for $|t| \ll \tau_c$, $D_Q(\epsilon)$ should be related to $a(\epsilon=0,t)$ as $D_Q(\epsilon) = a(\epsilon=0, \kappa/\epsilon^2)$. In this way the irreversibility is related to the pure ($\epsilon=0$) time evolution process. We note, however, that such a relation breaks down for small ϵ, because $a(\epsilon,\tau_c)=0$ for $\tau_c = \kappa/\epsilon^2 \gg T_s$ does not reproduce the perturbative result in the perturbative regime. This implies that $\kappa/\epsilon_T^2 \sim T_s$, i.e., $\epsilon_T \sim (\kappa/T_s)^{1/2}$.

The relation $D_Q(\epsilon) = a(\epsilon=0, \kappa/\epsilon^2)$ tells us that the pure time evolution should have the following properties to be consistent with the behaviors in the classical and quantum regimes: Up to the time $T_r = \kappa/\epsilon_C^2$ the moment $M_{\epsilon=0}(t) = \sum_{s=0}^{t} a(\epsilon=0,s)$ grows in agreement with the

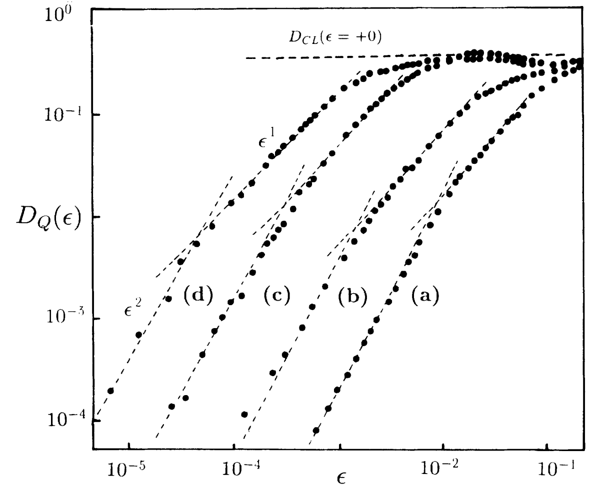

FIG. 2. Global characteristics of quantum irreversibility for various $\hbar/2\pi$. (a) $\hbar/2\pi = \frac{21}{512}$, (b) $\frac{21}{1024}$, (c) $\frac{21}{2048}$, and (d) $\frac{21}{4096}$ ($K=2.0$).

classical dynamics, i.e., $M_{\epsilon=0}(t) = D_{\rm CL}t$, and beyond T_r it increases according to a power law $M_{\epsilon=0}(t) \propto t^\beta$ with the exponent $\beta = 1 - \nu/2$. The former behavior agrees with the well-known result for the kicked rotor.[4,8] Thus T_r is the time scale on which the quantum-classical correspondence applies and is estimated to be $\ln\hbar/\lambda$.[1-3,8] Hence we obtain $\epsilon_C \sim (\kappa/T_r)^{1/2} \sim \hbar/(\delta p T_r^{1/2})$. However, the power growth for $T_r < t (< T_s)$ has not been previously reported. To verify this power growth we depict in Fig. 3 a typical transient evolution of $M_{\epsilon=0}(t)$ starting with a momentum eigenstate. Obviously, there are two time scales corresponding to T_r and T_s between which a power growth with the exponent β very close to

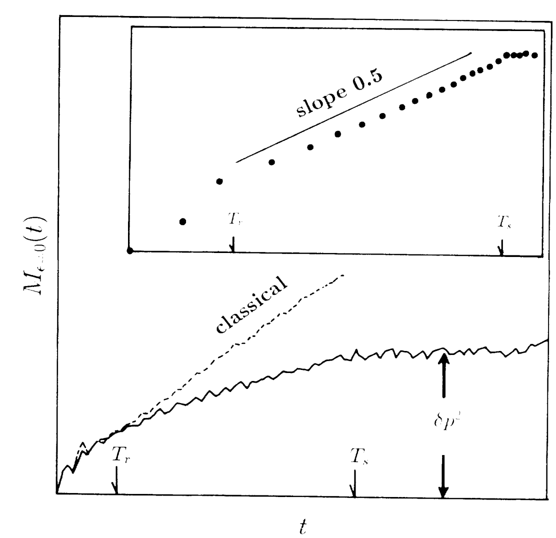

FIG. 3. Transient behavior of the moment $M_{\epsilon=0}(t) = \langle \Delta\hat{p}_t^2 \rangle$ for the pure ($\epsilon=0$) process. Inset: Log-log plot.

0.5 is observed. This predicts $\nu=2(1-\beta)\sim 1.0$, consistent with the value obtained for $D_Q(\epsilon)$ in the quantum regime. Thus the curious behavior in the quantum regime is attributed to the power growth in the pure quantum time evolution.

The second threshold ϵ_T stands for the convergence radius of the perturbation theory. It is evaluated with use of the fact that $D_Q(\epsilon)$ in the quantum regime must agree with the classical result $D_Q(\epsilon)=D_{\rm Cl}$ and the perturbative one $D_Q(\epsilon)\sim(\epsilon^2/\hbar^2)(\delta p)^4$ at $\epsilon=\epsilon_C$ and $\epsilon=\epsilon_T$, respectively. This yields

$$\epsilon_T/\epsilon_C=[D_{\rm Cl}T_r/(\delta p)^2]^{1/(2-\nu)}\sim\hbar/\delta p.$$

Seeing that $\epsilon_T/\epsilon_C\sim(T_r/T_s)^{1/2}$, the smallness of the ratio ϵ_T/ϵ_C is a reflection that the pure quantum evolution is governed by the two characteristic times T_r and T_s with quite different scales in the semiclassical limit.

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