Scaling, Diffusion, and the Integer Quantized Hall Effect

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The behavior of the two-particle spectral function, $S(q,\omega)$, is examined in the hydrodynamic regime, at the mobility edge in a model for the integer quantized Hall effect. Results are presented from numerical diagonalization of the Hamiltonian for finite systems. For q^2/ω small, $S(q,\omega)$ has a conventional, diffusive form. For q^2/ω large, the novel dependence $S(q,\omega)\sim\omega^{-\eta/2}q^{-2+\eta}$ is obtained, with $\eta=0.38\pm0.04$.

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Scaling behavior near a mobility edge—viewed as a critical point—is widely believed to be especially simple, at least in the absence of magnetic effects and interactions.[1] This simplicity is a consequence of the one-parameter scaling assumption,[1] that a length-dependent dimensionless conductance is the only quantity necessary to characterize behavior at a given length scale. As a result there is only one independent critical exponent say the localization-length exponent ν, which describes the approach to the mobility edge and from which other exponents, for example, the one for conductivity, can be determined. Correspondingly, at the mobility edge itself, the requirement of homogeneity under combined length and energy scale transformations determines the scaling form of eigenfunction correlations completely.[1] One implication is that the exponent η, governing correlations at the mobility edge, should be given exactly in d dimensions by $\eta=2-d$.

Observation of the integer quantized Hall effect indicates a breakdown of one-parameter scaling. Pruisken and co-workers[2] have shown how this happens within an effective field theory: The Hall conductance appears as a second coupling constant, in addition to the dissipative conductance, and two-parameter scaling emerges.

We describe in this Letter a distinct and striking aspect of the failure of one-parameter scaling at the mobility edge in a model for the integer quantum Hall effect. Eigenfunction correlations do not have a conventional, diffusive form: Although homogeneous, they are characterized by a nontrivial value of η. Equivalently, the diffusion constant (proportional to the dissipative conductance) has a novel wave-vector and frequency dependence: Eq. (2).

Our main results are from numerical diagonalization of the model Hamiltonian. Before describing these calculations, we introduce the quantity studied, the two-particle spectral function, and summarize scaling arguments, our conclusions, and previous analytical work.

The two-particle spectral function is defined by

$$S(\mathbf{r};E,\omega)=\left\langle\sum_{\alpha,\beta}\delta(E-\omega/2-E_\alpha)\delta(E+\omega/2-E_\beta)\psi_\alpha(\mathbf{0})\psi_\alpha^*(\mathbf{r})\psi_\beta(\mathbf{r})\psi_\beta^*(\mathbf{0})\right\rangle,$$

where $\psi_\alpha(\mathbf{r})$ and $\psi_\beta(\mathbf{r})$ are eigenfunctions with energies E_α and E_β and the angular brackets denote an average over an ensemble of disordered systems. It, or its Fourier transform $S(q;E,\omega)$, determines the system's linear response to time- and space-dependent variations of the chemical potential. The connection between the spectral function and the retarded-advanced two-particle Green's function, and a Ward identity[3] for the latter, restrict the form for small q,ω to be

$$S(q;E,\omega)=\rho(E)h(q,\omega)/\pi[\omega^2+h^2(q,\omega)]$$

with $h(q=0,\omega)=0$, where $\rho(E)$ is the density of states in energy, per unit d-dimensional volume.

If a wave packet, constructed as a superposition of eigenstates close in energy to E, spreads diffusively for long times and for large distances with diffusion constant D, then $h(q,\omega)=\hbar Dq^2$ for small q,ω. More generally, if $E=E_c$, the mobility edge energy, then $h(q,\omega)$ should satisfy a homogeneity requirement. In a system of linear size L, one expects, over lengths much greater than the elastic scattering length, l_{elastic}, that L itself sets the unit of length. Correspondingly, over energies ω much less than the bandwidth, the mean level spacing, which is proportional to L^{-d}, sets the energy scale. Invariance under the scale change $L\to bL$ implies[1]

$$h(q,\omega)=b^d h(b^{-1}q,b^{-d}\omega)\equiv\hbar q^d f(q^d/\omega). \tag{1}$$

The new results presented in this paper are that, for electrons in the lowest Landau level with disorder, $f(q^2/\omega)$ has the form

$$f\left[\frac{q^2}{\omega}\right]=\begin{cases}D, & \text{if } q^2/\omega<c\rho(E_c),\\ [\omega c\rho(E_c)/q^2]^{\eta/2}D, & \text{if } q^2/\omega>c\rho(E_c),\end{cases} \tag{2}$$

for q,ω small. Estimated parameter values are given after Eq. (5). [Presumably there is a crossover region between the two limiting forms of Eq. (2), but apparent-

ly it is rather narrow.]

Thus an ω-independent characteristic length, $L_0 = [\omega c \rho(E_c)]^{-1/2}$, is generated, which can be interpreted as the size of a system having a mean level spacing at the mobility edge of order ω. Long-wavelength eigenfunction correlations, $\xi^{-1} < q < L_0^{-1}$, where ξ is the localization length, have the conventional, diffusive form, but at shorter wavelengths, $L_0^{-1} < q < l_{\text{elastic}}^{-1}$, correlations have the novel behavior $S(q;E_c,\omega) \propto \omega^{-\eta/2} q^{-2+\eta}$.

Our numerical calculations were motivated by recent analytical results.[4,5] First, a rigorous inequality for eigenfunction correlations in a single Landau level[4] excludes conventional, diffusive behavior for $S(q,\omega)$, but is consistent with the modified form we propose. (The correlation inequality *would* be satisfied with $\eta = 0$ *if* $\nu \lesssim \frac{1}{2}$, but this range for ν is ruled out by finite-size scaling calculations[6] and the Harris-Mott criterion.[7]) Second, perturbative calculations of quantum interference effects in high Landau levels,[5] analogous to weak-localization calculations in small magnetic field, reveal that the leading correction to diffusive behavior is a reduction in the diffusion constant at finite wave vector. This correction arises at first order in perturbation theory and therefore dominates over the universal reduction in dissipative conductivity found at second order. We interpret this result to indicate a crossover with increasing length scale from diffusive ($\eta = 0$) to critical ($\eta > 0$) correlations.

Our numerical results are for electrons in the lowest Landau level, moving in a Gaussian white-noise potential. We start from the Hamiltonian for noninteracting particles of mass m and charge $-|e|$ moving in the x-y plane with a uniform, perpendicular magnetic field of strength B and a potential $V(x,y)$:

$$H = (\hbar\omega_c/2)\{-l_c^2 \partial^2/\partial x^2 + (-l_c \partial/\partial y + x/l_c)^2\} + V(x,y) \equiv H_0 + V(x,y), \tag{3}$$

where $l_c^2 = \hbar/|e|B$ and $\omega_c = eB/m$. We treat a square system of side L and apply periodic boundary conditions in both directions, which requires $L^2 = 2\pi N l_c^2$, N integer. Eigenstates of H_0 from the lowest Landau level are then products of θ functions,[8] which are accurately approximated for the system sizes we study ($128 \le N \le 1024$) by

$$\langle x,y|m\rangle = \frac{(2N)^{1/4}}{L} \exp\left\{\frac{2\pi i m}{L}[y] - \frac{1}{2l_c^2}\left[x + \frac{mL}{N}\right]^2\right\},$$

for $m = 1,2,\ldots,N$, with $m + N \equiv m$, where square brackets indicate lengths measured modulo L, so that $-L/2 \le [x + mL/N] < L/2$ and $0 \le [y] < L$. In the strong-field limit, scattering between Landau levels may be neglected; the projection of Eq. (3) onto the lowest level is, after subtraction of zero-point energy,

$$\mathcal{H} = \sum_{m,m'=1}^{N} |m\rangle\langle m|V(x,y)|m'\rangle\langle m'|.$$

We take the potential, $V(x,y)$, to be essentially Gaussian white noise with zero mean and covariance

$$\langle V(x,y)V(x',y')\rangle = v^2 l_c^2 \delta(x-x')\delta(y-y').$$

More precisely, we take the Fourier components of $V(x,y)$ with wave vectors (k_x,k_y) for which $|k_x|$ or $|k_y| > 6.3 l_c^{-1}$ are omitted. Their contributions to the matrix elements of $\mathcal{H}$ are negligible, since the basis states are smooth functions.

We calculate $S(q;E_c,\omega)$ by numerically diagonalizing $\mathcal{H}$, assembling the eigenvectors and energies to form the spectral function, averaging over different realizations of the potential and, finally, extrapolating from a sequence of system sizes to obtain the large-N, small-q, small-ω limit.[9] If the eigenvector of $\mathcal{H}$ with energy E_α has expansion coefficients $a_\alpha(m)$ in the basis $\{|m\rangle\}$, let

$$Q_{\alpha\beta}(k,l) = N\left|\sum_{m=1}^{N} a_\alpha(m) a_\beta^*(m+l)\, e^{2\pi i k m/N}\right|^2,$$

with k,l integer. Then[4]

$$S(q;E,\omega) = (2\pi l_c^2 N^2)^{-1} e^{-q^2 l_c^2/2} \left\langle \sum_{\alpha,\beta} \delta(E - \omega/2 - E_\alpha)\delta(E + \omega/2 - E_\beta) Q_{\alpha\beta}(k,l) \right\rangle, \tag{4}$$

where $q^2 = 2\pi(k^2 + l^2)/N l_c^2$.

This expression is unsuitable for numerical evaluation because of the two δ functions. One is removed by integration on E over a narrow range around the mobility edge, and we replace the other with a sharply peaked weighting function. Thus we assume that $S(q;E,\omega)$ is independent of E over the range of energies for which the localization length in an infinite system is much larger than the size of our system. From the work of Aoki and Ando[6,10] we estimate this condition to be satisfied generously for the $2M$ eigenstates closest to $E = E_c \equiv 0$, if $M \le 0.35 N^{3/4}$. Equation (4) may then be replaced by

$$S(q;E,\omega) = 2\pi l_c^2 \rho^2(0) \exp(-q^2 l_c^2/2) K(q,\omega),$$

with

$$K(q,\omega) = \left\langle {\sum}'_{\alpha,\beta} w(\omega + E_\alpha - E_\beta) Q_{\alpha\beta}(k,l) \right\rangle \Big/ \left\langle {\sum}'_{\alpha,\beta} w(\omega + E_\alpha - E_\beta) \right\rangle, \tag{5}$$

where $\sum'_{\alpha,\beta}$ is the sum over the $2M$ states described above[11] and $w(\omega+E_\alpha-E_\beta)$ is chosen to be a Gaussian function with width σ of order the mean level spacing: $\sigma=\sigma_0/N$, $\sigma_0=0.64v$.

In this way we obtain $K(q,\omega)$ for $0\le k^2+l^2\le 19$ and $\omega=n\sigma$, $2\le n\le 19$ (k,l,n integer), where the upper limits are chosen so that $\omega\ll v$ and $q<l_c^{-1}$, even in the smallest systems studied. Averaging over 1000 realizations for $N=128$, 200 for $N=256$, 50 for $N=512$, and 10 for $N=1024$ resulted in statistical uncertainties of approximately 1% in $K(q,\omega)$.

We make the extrapolations $N\to\infty$ and $q,\omega\to 0$, with q^2/ω fixed, simultaneously as follows. If the scaling assumption, Eq. (1), is correct, then in these limits we expect

$$\omega K(q,\omega)=[2\pi^2 l_c^2\rho(0)]^{-1}[\hbar q^2 f(q^2/\omega)/\omega]/[1+\{\hbar q^2 f(q^2/\omega)/\omega\}^2],$$

which is a function only of the single variable q^2/ω. Each system size studied results in values for $\omega K(q,\omega)$ at twelve values of $q^2=2\pi(k^2+l^2)/Nl_c^2$, and eighteen values of $\omega=n\sigma_0/N$, producing 140 distinct values of q^2/ω. For each q^2/ω we combine data from all system sizes and extrapolate $\omega K(q,\omega)$ linearly in q^2 to $q=0$ and, by implication, $N=\infty$, $\omega=0$.[12] This extrapolation is illustrated in Fig. 1 for four representative q^2/ω values.

A preliminary examination of the extrapolated $\omega K(q,\omega)$, and hence $S(q;0,\omega)$, shows that a form similar to Eq. (2) is appropriate. Best values of $\rho(0)$, D, c, and η are obtained as follows. The $\omega K(q,\omega)$ values are divided into two sets: those for $q^2/\omega<a$ and those for $q^2/\omega>a$. The first set is fitted with $f(q^2/\omega)=D$ and the second set with $f(q^2/\omega)=(\omega a/q^2)^{\eta/2}D$, adjusting $\rho(0)$, D, a, and η. In this way we find $vl_c^2\rho(0)=0.149$, $\hbar D/vl_c^2=0.58$, $c\equiv a/\rho(0)\simeq 60$. The exponent lies in the range $\eta=0.38\pm 0.04$. Error estimates are problematic, both because of statistical correlations contained in $K(q,\omega)$ and because of uncertainty over systematic effects.

The value for $vl_c^2\rho(0)$ compares well with the known, exact result,[13] $\sqrt{2}/\pi^2\simeq 0.143$. The present value of the long-wavelength diffusion constant agrees reasonably well with a previous calculation[14] from a resummed perturbation series in v, of $\hbar D/vl_c^2=0.51$. At critical points in two-dimensional, statistical-mechanical systems, there is an established relation between η and a finite-size scaling amplitude.[15] If this relation also holds at a mobility edge, η can be found independently[16] from transfer matrix calculations of the localization length[16,17]; in fact, such estimates are $\simeq 30\%$ larger than the value given above.

A generalization of the functional form that we fitted is obtained by allowing a second exponent, $\eta'\neq 0$, for $q^2/\omega<a$. From this we obtain $\eta'=0.064$, with little change in η. Since the zero-temperature, dc dissipative conductivity is proportional to $f(0)$, $\eta'\equiv 0$ is clearly preferred.

The behavior of $f(q^2/\omega)$ is shown in Fig. 2. The collapse of the data onto a single curve is strong support for the scaling assumption, Eq. (1). It is clear that eigenfunction correlations near the mobility edge, and on length scales between the magnetic length and the localization length, cannot be characterized solely by a diffusion constant; instead they are well represented by Eq. (2).

Our results give no direct insight into the reason for the divergence of the localization length at the Landau band center. However, we feel that a complete theory of localization in the integer quantum Hall regime should

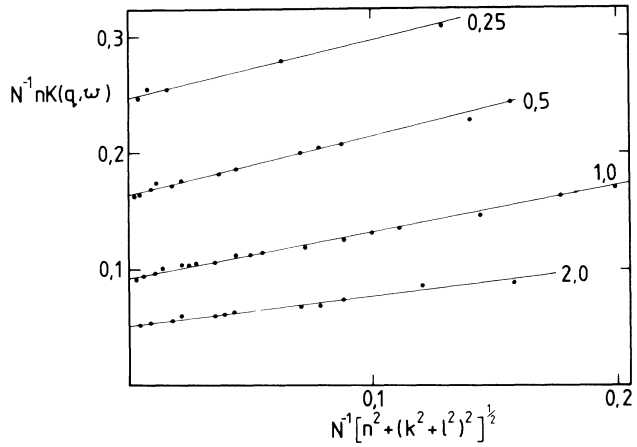

FIG. 1. Illustration of extrapolation of $K(q,\omega)$ to $N=\infty$, $q^2=\omega=0$ with q^2/ω fixed. For clarity, only a representative selection of data points are shown. Lines are labeled with values of $(k^2+l^2)/n$.

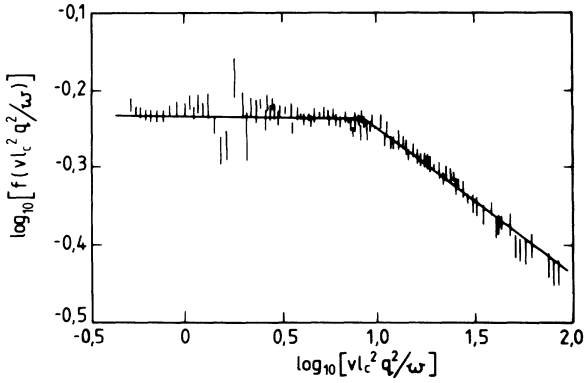

FIG. 2. Fit of extrapolated data (points with estimated errors) by Eq. (2) (line), for parameter values given in text. A representative selection of the 140 data points is shown. The large errors near $\log_{10}(vl_c^2q^2/\omega)=0.2$ reflect the intrinsic difficulty in our obtaining $f(q^2/\omega)$ from $S(q,\omega)$ near the maximum in $S(q,\omega)$. Units of f are $vl_c^2/\hbar$.

include predictions of the scaling form Eq. (2), for eigenfunction correlations on scales shorter than the localization length. The behavior we find for small q^2/ω is compatible with the field theory proposed by Pruisken and co-workers[2]; the anomalous correlations at large q^2/ω may have an instanton interpretation in that context.

A direct experimental test of the scaling form is likely to be difficult because the obvious measurement, conductivity, probes $q=0$. An indirect signature should appear in the temperature dependence of the inelastic scattering rate, and hence the peak values of the dissipative conductivity between Hall plateaus, because the spatial overlap of eigenfunctions close in energy is large:

$$\langle |\psi_\alpha(\mathbf{r})\psi_\beta(\mathbf{r})|^2\rangle \sim |E_\alpha - E_\beta|^{-\eta/2}$$

for E_α, E_β near E_c. In theoretical terms, the consequence of the eigenfunction correlations we find is that a wave packet spreads with (diameter)$^2 \propto$ time, because the scaling variable is q^2/ω, but the wave packet never approaches the Gaussian asymptotic form reached by solutions to the diffusion equation. Clearly, the nonzero value of η may be a consequence of fractal structure in eigenfunctions.[18]

It is natural to ask whether similar correlations are likely to occur near the mobility edge in other systems. An obvious candidate is the localization transition in two dimensions that occurs when spin-orbit scattering is strong.

In summary, we have shown that eigenfunction correlations have novel scaling behavior near the mobility edge in a model for the integer quantum Hall effect.

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[9]Diagonalization of a square system is a suitable technique because we are interested in properties on scales short compared with the (divergent) localization length; to determine the localization length itself, finite-size scaling calculations on long strips (Ref. 6 and references therein) would of course be more appropriate. Matrix diagonalization used routines supplied by the Nottingham algorithms group.

[10]The authors of Ref. 6 use for the potential $V(x,y)$ a high concentration of mixed repulsive and attractive short-range scatterers, which is distinct from, but close to, our choice of Gaussian white noise.

[11]We omit diagonal terms, $\alpha=\beta$, from the sums in Eq. (5), which would otherwise make a small contribution to $K(q,\omega)$ via the tails of $w(\omega+E_\alpha-E_\beta)$. This is done because the orthogonality of eigenstates results in behavior for $Q_{\alpha\alpha}(k,l)$ different from that for $Q_{\alpha\beta}(k,l)$, $\alpha\neq\beta$.

[12]The physical limit is actually *first* $N\to\infty$, *then* $q^2,\omega\to 0$. It is not feasible to reach the regime $l_c^{-1}\gg q\gg L^{-1}$ with the present technique. If results did depend on the value of Lq, points in Fig. 1 would not like on single curves for each value of q^2/ω.

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