Canonical Quantum Field Theory with Exotic Statistics

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An explicit example of a three-dimensional quantum field theory with exotic statistics and fractional spin is presented.

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If particles are not allowed to have coinciding positions, the configuration space of a two-dimensional gas has nontrivial first homotopy group. As a result the many-body wave functions, which must be single valued on the covering space, can be multivalued functions on the configuration space itself. This allows wave functions for identical particles to change by a phase when particle positions are interchanged and it is therefore possible that two-dimensional quantum mechanics exhibits many-particle states with exotic exchange statistics.[1-5] Intimately related to exotic statistics is fractionally quantized angular momentum. The spin generates the covering group R^1 of the rotation group SO(2) and can therefore have fractional eigenvalues.

Fractional statistics have been argued to play a role in several phenomena in condensed matter where dynamics are effectively confined to a plane. Most notable are the fractionally quantized Hall effect[6] and the behavior of vortices in superfluid helium films.[7] They have also been conjectured to be of importance in two-dimensional antiferromagnets and in the magnetic properties of the layered copper-oxide materials which exhibit high-temperature superconductivity.[8-10] In this Letter I shall examine a concrete realization of exotic spin and statistics. The idea is to modify a three-spacetime-dimensional field theory which has a conserved U(1) current by the addition of minimal coupling to an Abelian gauge field and a Chern-Simons term[11]:

$$S=S_{\text{matter}}+\int d^3x\left[A_\mu j^\mu+\frac{\alpha}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\right]. \tag{1}$$

Previous evidence for fractional statistics in this model comes from spacetime arguments: For the currents due to the motion of classical charged particles the effective action obtained by the solving of the equation of motion for the gauge field in (1),

$$S_{\text{eff}}=\frac{\pi}{\alpha}\int\epsilon^{\mu\nu\lambda}j_\mu\frac{\partial_\nu}{\partial^2}j_\lambda,$$

gives the linking number of the trajectories.[10] For a single particle the effective action is the torsion of the trajectory which, for $\alpha=1$, Polyakov[10] has argued can be represented by fermionic variables. Furthermore, the modification (1) of the O(3) nonlinear σ model gives the Hopf term which is related to the linking number of soliton trajectories.[2,3] In the following I show that (1) is equivalent to the model without the gauge fields and with multivalued charged operators which have graded commutators and create quantum states with fractional statistics. Let us begin with the prototype of a charged scalar

$$S=\int d^3x[(\partial_\mu+iA_\mu)\phi^*(\partial^\mu-iA^\mu)\phi-m^2\phi^*\phi+(\alpha/4\pi)\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda]. \tag{2}$$

To implement canonical quantization we identify the momenta

$$\pi=(\partial_0-iA_0)\phi,\quad \pi^*=(\partial_0+iA_0)\phi^*, \tag{3}$$

$$\pi_0\approx 0,\quad \pi_i-(\alpha/4\pi)\epsilon_{ij}A_j\approx 0. \tag{4}$$

Here and in the following the modified equal sign ($\approx$) indicates a constraint, i.e., a relation which eliminates dynamical variables from the phase space. The Hamiltonian is

$$H=\int d^2x\{\pi^*\pi+\phi^*(\overleftarrow{\boldsymbol{\partial}}+i\mathbf{A})\cdot(\overrightarrow{\boldsymbol{\partial}}-i\mathbf{A})\phi+m^2\phi^*\phi-A_0[j_0+(\alpha/2\pi)B]\}, \tag{5}$$

where $j_0=i(\phi^*\pi-\pi^*\phi)$ is the charge density. Conserving the first-class constraint $\pi_0\approx 0$ yields the secondary constraint

$$j_0+(\alpha/2\pi)B\approx 0, \tag{6}$$

where $B=\epsilon_{ij}\partial_iA_j$ is the magnetic field. This constraint is responsible for fractional statistics. With (4) we recognize

(6) as the generator of static gauge transformations. Gauge invariance therefore requires that magnetic flux is associated with charge density wherever the latter is nonzero. Bohm-Aharonov phase factors for the motion of charges in each other's magnetic fields account for the changes of phase of wave functions as trajectories of particles link with each other. We complete the constraints (4) and (6) by imposing the gauge conditions $A_0 \approx 0$, $\boldsymbol{\partial}\cdot\mathbf{A} \approx 0$. These, together with (4) and (6) determine the gauge field:

$$\hat{A}_i(\mathbf{x}) = \frac{1}{\alpha}\epsilon_{ij}\int d^2y \frac{x_j - y_j}{(\mathbf{x}-\mathbf{y})^2} j_0(\mathbf{y}) \tag{7}$$

and the Hamiltonian is given by (5) with A_0 set to 0, $\mathbf{A}$ set to $\hat{\mathbf{A}}$ and with the canonical Poisson brackets $\{\phi(\mathbf{x}),\pi^*(\mathbf{y})\} = \{\phi^*(\mathbf{x}),\pi(\mathbf{y})\} = \delta^2(\mathbf{x}-\mathbf{y})$. In the quantum theory $\hat{\mathbf{A}}$ and $\hat{\phi}$ do not commute and we must specify an operator ordering for the Hamiltonian. We use the convention already implicit in (5) that covariant derivatives of ϕ,π operate from the right and of ϕ^*,π^* operate from the left. The gauge field (7) is the divergence of a multivalued operator,

$$\hat{\mathbf{A}}(\mathbf{x}) = -\boldsymbol{\partial}(i/\alpha)\int d^2y\, \Theta(\mathbf{x}-\mathbf{y}) j_0(\mathbf{y}),$$

where $\Theta(\mathbf{x}-\mathbf{y})$ is the angle between the vector $\mathbf{x}-\mathbf{y}$ and the x_1 axis. The interaction terms can be removed from the Hamiltonian by the gauge transformation

$$\hat{\phi}(\mathbf{x}) = \exp\left[(i/\alpha)\int d^2y\, \Theta(\mathbf{x}-\mathbf{y}) j_0(\mathbf{y})\right]\phi(\mathbf{x}),$$
$$\hat{\pi}(\mathbf{x}) = \exp\left[(i/\alpha)\int d^2y\, \Theta(\mathbf{x}-\mathbf{y}) j_0(\mathbf{y})\right]\pi(\mathbf{x}), \tag{8a}$$

$$\hat{\phi}^*(\mathbf{x}) = \phi^*(\mathbf{x})\exp\left[-(i/\alpha)\int d^2y\, \Theta(\mathbf{x}-\mathbf{y}) j_0(\mathbf{y})\right],$$
$$\hat{\pi}^*(\mathbf{x}) = \pi^*(\mathbf{x})\exp\left[-(i/\alpha)\int d^2y\, \Theta(\mathbf{x}-\mathbf{y}) j_0(\mathbf{y})\right]. \tag{8b}$$

The Hamiltonian takes the form

$$H = \int \{\hat{\pi}^*\hat{\pi} + \hat{\phi}^*\overleftarrow{\boldsymbol{\partial}}\cdot\overrightarrow{\boldsymbol{\partial}}\hat{\phi} + m^2\hat{\phi}^*\hat{\phi}\}, \tag{9}$$

and $\hat{\phi}$ and $\hat{\pi}$ obey the graded equal-time commutation relations,

$$\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y}) - e^{-(i/\alpha)\Delta}\hat{\phi}(\mathbf{y})\hat{\phi}(\mathbf{x}) = 0, \quad \hat{\phi}^*(\mathbf{x})\hat{\phi}(\mathbf{y}) - e^{(i/\alpha)\Delta}\hat{\phi}(\mathbf{y})\hat{\phi}^*(\mathbf{x}) = 0, \tag{10a}$$

$$\hat{\phi}(\mathbf{x})\hat{\pi}^*(\mathbf{y}) - e^{-(i/\alpha)\Delta}\hat{\pi}^*(\mathbf{y})\hat{\phi}(\mathbf{x}) = i\delta^2(\mathbf{x}-\mathbf{y}), \quad \hat{\phi}(\mathbf{x})\hat{\pi}(\mathbf{y}) - e^{-(i/\alpha)\Delta}\hat{\pi}(\mathbf{y})\hat{\phi}(\mathbf{x}) = 0, \tag{10b}$$

with $\Delta = \Theta(\mathbf{x}-\mathbf{y}) - \Theta(\mathbf{y}-\mathbf{x}) = \pi + 2\pi$ (integer). Multivaluedness of the phase Δ is essential to consistency of the commutation relations (10). Many-particle states created by repeated operation of $\hat{\phi}(\mathbf{x}_i),\hat{\phi}^*(\mathbf{y}_j)$ are multivalued functions of $\{\mathbf{x}_i\},\{\mathbf{y}_j\}$. [The products of operators in (8) are singular and must be defined with a regularization such as point splitting. In general, the commutators in (10) are insensitive to the regularization.] The equation of motion for $\hat{\phi}$ is $\partial_\mu\partial^\mu\hat{\phi} = 0$. This indicates a dual description of the field theory (2): (i) single-valued fields with complicated interactions and Hamiltonian (5); (ii) free fields with multivalued operators and explicit fractional statistics.

To examine the connection between spin and statistics, consider the gauge-invariant symmetric energy-momentum tensor

$$T_{\mu\nu} = (\partial_\mu + iA_\mu)\phi^*(\partial_\nu - iA_\nu)\phi + (\partial_\nu + iA_\nu)\phi^*(\partial_\mu - iA_\mu)\phi - g_{\mu\nu}[(\partial_\lambda + iA_\lambda)\phi^*(\partial^\lambda - iA^\lambda)\phi - m^2\phi^*\phi]. \tag{11}$$

The integrated energy density $\int T_{00}$ is the Hamiltonian (5). The momentum density is $T_{0i} = \pi^*\partial_i\phi + \partial_i\phi^*\pi + A_i j_0$. This gauge-invariant operator generates a translation of the gauge-variant field ϕ accompanied by an A_i-dependent gauge transformation so that in the classical field theory the resulting transformation is gauge invariant $\delta_i\phi(\mathbf{x}) = \{\phi(\mathbf{x}),\int d^2y\, T_{0i}(\mathbf{y})\} = (\partial_i - iA_i)\phi(\mathbf{x})$. Similarly, the angular momentum operator $L = \int d^2x\, x_i\epsilon_{ij}T_{0j}$ generates a gauge-invariant rotation of ϕ. On the constrained classical phase space we replace $\mathbf{A}$ by $\hat{\mathbf{A}}$:

$$P_i = \int d^2x(\pi^*\partial_i\phi + \partial_i\phi^*\pi) + \frac{1}{\alpha}\int d^2x\, d^2y\, j_0(\mathbf{x})\epsilon_{ij}\frac{(\mathbf{x}-\mathbf{y})_j}{(\mathbf{x}-\mathbf{y})^2}j_0(\mathbf{y}) = \int d^2x(\pi^*\partial_i\phi + \partial_i\phi^*\pi), \tag{12}$$

$$L = \int d^2x(\pi^*\mathbf{x}\times\boldsymbol{\partial}\phi + \mathbf{x}\times\boldsymbol{\partial}\phi^*\pi) + \frac{1}{\alpha}\int d^2x\, d^2y\, j_0(\mathbf{x})\frac{\mathbf{x}\cdot(\mathbf{x}-\mathbf{y})}{(\mathbf{x}-\mathbf{y})^2}J_0(\mathbf{y})$$
$$= \int d^2x(\pi^*\mathbf{x}\times\boldsymbol{\partial}\phi + \mathbf{x}\times\boldsymbol{\partial}\phi^*\pi) + Q^2/2\alpha, \tag{13}$$

where $Q = \int j_0$. In (12) and (13) we have used the symmetry of the integration in the last terms. When the constraints are imposed, the covariant momentum is identical to the canonical translation generator, whereas the covariant angular momentum is the sum of the canonical angular momentum and a term proportional to the square of the charge. We interpret the latter term as a spin operator. It cannot be removed by redefinition of the angular momentum since it is required that L satisfies the Lorentz algebra $\{K_i,K_j\} = \epsilon_{ij}L$, where K_i is the boost generator. At the classical level, ϕ and π have operator-valued spin, $\delta\phi = \{\phi,L\} = \mathbf{x}\times\boldsymbol{\partial}\phi + (Q/\alpha)\phi$. This anomalous spin was previously noticed by Hagen[11] who also correctly observed that in the corresponding quantum field theory the operators ϕ, ϕ^*, π, and π^* create single-valued states.

In the quantum theory we must take into consideration the ambiguous ordering of operators in T_{0i}, as well as singularities of operator products. In general this produces additional terms in the quantum versions of (12) and (13). If we take the same operator ordering convention in T_{0i} as we used for the Hamiltonian, then in terms of $\hat{\phi}$ and $\hat{\pi}$ the momentum, angular momentum, and boost operators are

$$\mathbf{P}=\int(\hat{\pi}^*\,\boldsymbol{\partial}\hat{\phi}+\boldsymbol{\partial}\hat{\phi}^*\,\hat{\pi}),$$
$$L=\int(\hat{\pi}^*\mathbf{x}\times\boldsymbol{\partial}\hat{\phi}+\mathbf{x}\times\boldsymbol{\partial}\hat{\phi}^*\,\hat{\pi}), \tag{14}$$
$$\mathbf{K}=\int\mathbf{x}(\hat{\pi}^*\hat{\pi}+\hat{\phi}^*\overleftarrow{\boldsymbol{\partial}}\cdot\overrightarrow{\boldsymbol{\partial}}\hat{\phi}+m^2\hat{\phi}^*\hat{\phi}).$$

With the graded commutation relations (10) and the Hamiltonian (9) these obey the Poincaré algebra. Furthermore $\hat{\phi}$ represents the rotations faithfully $e^{i\theta L}\hat{\phi}(\mathbf{x})\times e^{-i\theta L}=\hat{\phi}(\Lambda(\theta)\mathbf{x})$. The state $\prod_1^m\hat{\phi}(\mathbf{x}_i)\prod_1^n\hat{\phi}(\mathbf{y}_j)\times|\text{vacuum}\rangle$ is multivalued with canonical scalar spin parity. (Spin parity is the change in phase of a quantum state under rotation by 2π.) In terms of the complex variable, $z_i=x_{i1}+ix_{i2}$, the multivaluedness can be characterized by the familiar form[6]

$$\prod_1^m\hat{\phi}(\mathbf{x}_i)\,|\text{vacuum}\rangle$$
$$=\prod_{i<j}(z_i-z_j)^{1/\alpha}\times(\text{single-valued state}).$$

Alternatively, the state $\prod_1^m\phi(\mathbf{x}_i)\prod_1^n\phi^*(\mathbf{y}_j)\,|\text{vacuum}\rangle$ is single valued but has fractional spin parity $-(\pi/\alpha)\times(m-n)(m-n+1-2q_0)$, where q_0 is the vacuum charge. Also

$$e^{iL\theta}\phi(\mathbf{x})e^{-iL\theta}=e^{i(Q/\alpha\theta}\phi(\Lambda(\theta)\mathbf{x})$$

is a projective representation of the rotations with operator-valued phase.

The present results can be applied to field theories with other types of matter-nonrelativistic Schrödinger fields or fermions. Another interesting application is to the O(3) nonlinear σ model where in (1) we would set $S_{\text{matter}}=\int\frac{1}{2}\,\partial_\mu n^a\partial^\mu n^a$ with $\sum_1^3 n^a n^a=1$ and take

$$j^\mu=(1/8\pi)\epsilon^{\mu\nu\lambda}\epsilon_{abc}n^a\,\partial_\nu n^b\,\partial_\lambda n^c$$

as the topological current. The solving of the equation of motion for the gauge field in (1) leads to an effective action which is the nonlocal Hopf invariant with coefficient $1/2\alpha$. It is already known[5] that the angular momentum in this case has the same form as (12), $L=L_{\text{canonical}}+Q^2/2\alpha$, where Q is the soliton number which is quantized as integers. A single-valued operator $U(\mathbf{x})$ which creates a soliton $\mathbf{x}$ has fractional (operator-valued) spin. (Since $[Q,U]=-U$.) The multivalued operator $\exp[i/\alpha\int\Theta j^0]U(\mathbf{x})$ creates a soliton with a multivalued wave function and is a rotation scalar. It would be interesting to develop a theory of quantized solitons with fractional statistics from this point of view.

The present construction can be viewed as progress toward a generalization to three spacetime dimensions of the boson-fermion equivalence[12] in two dimensions where creation operators for solitons are used to obtain states with Fermi statistics in a purely bosonic field theory. It is also in line with the ideas of Rajeev,[13] who argued that nontrivial projective representations of four-spacetime-dimensional current algebra are responsible for the Fermi statistics of skyrmions. Vertex operators which have graded commutation relations in two space dimensions have previously been constructed by Murthy.[14] Though the precise relationship of his operators with $\hat{\phi},\hat{\pi}$ is as yet unclear, my construction can be regarded as a concrete realization of some of those ideas. It remains for us to examine the renormalization of the model (2). Since α is dimensionless the interaction introduced by the Chern-Simons term is strictly renormalizable. In particular, it would be interesting to estimate the α dependence of the partition function.

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