Average versus Typical Mean First-Passage Time in a Random Random Walk

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Random walk in a one-dimensional random medium of length N is analyzed. It is rigorously shown that in most realizations of the medium, the mean first-passage time, $\bar{t}$, bears the following relation to N, for large N: $\log\bar{t} \propto \sqrt{N}$. The average of $\bar{t}$ over the realizations of the medium, $\langle t\rangle$, satisfies $\log\langle t\rangle \propto N$. Our formalism, though being exact, employs only elementary means and makes transparent the physics of the delay experienced by the random walker: It is due to the existence of subsegments in which the bias against motion towards the desired end is largest. Some implications of these results concerning the replica method are briefly discussed.

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The transport properties of random media are crucially different from those in homogeneous or periodic matter.[1] A large portion of the transport properties in random materials is usually coined anomalous.[2] There is no general theory for anomalous transport, perhaps because no single theory can encompass the full richness of the phenomena involved. Special attention has been drawn to one-dimensional models.[3-6] The latter can be used as mathematical constructs exhibiting anomalous transport; they are useful in modeling higher-dimensional dilute networks. An interesting one-dimensional problem, known as the Sinai problem,[7] has been considered by many investigators.[7-11] The problem is one of a random (discrete) walk on a one-dimensional lattice, $-\infty < j < \infty$ (j being the integers). At each site, j, there is a probability p_j to hop to site $j+1$ per (discrete) time unit and a probability $q_j = 1-p_j$ to hop to site $j-1$. The set $\{p_j\}$ consists of independent random variables satisfying $0 < p_j < 1$. A probability distribution for the values of $\{p_j\}$ is defined so that $\log(p_j/q_j)$ has zero mean (i.e., no average bias) and a finite variance σ^2. A realization would be a choice of the values of the p_j's. Various methods[8-11] have been used to show that the mean-square displacement of a walker in such a system is proportional to $(\log t)^4$, t being the time. The probability distribution of such a walk has also been considered.[7]

In the present Letter, we consider the mean first-passage time (MFT), $\bar{t}$, from site $j=0$ to site $j=N$. The hopping probabilities p_j are defined as before except at $j=0$, where $q_0 = 1-p_0$ is a waiting probability per time unit. We set out to show that $\log\bar{t} \propto \sqrt{N}$ for typical realizations, whereas the realization averaged MFT, $\langle t\rangle$, satisfies $\log\langle t\rangle \propto N$ for $N \gg 1$. Thus averaged quantities are atypical. The possibility of nonself averaging in statistical systems has been realized before.[12]

The method to be used below has been described in detail in the literature[13,14] (an alternative method,[15] based directly on the solution of the master equation describing the above random walk yields equivalent results). Assume a given realization of the set $\{p_j\}$. Define the following probability distribution functions[14]: (1) $\hat{T}_{i,j}(n)$: the probability to leave site i on the first step and reach site j, for the first time, following n time units, in which i was never revisited. (2) $\hat{Q}_{i,j}(n)$: the probability to move from i to $i+1$ on the first step and return to i, for the first time, after n time units, without ever reaching j. (3) $\hat{G}_{i,j}(n)$: the probability to leave i and reach j, for the first time following n time units (returning to i is allowed). The generating function corresponding to any probability distribution function, $\hat{D}(n)$, is defined as $D(z) = \sum_{n=0}^{\infty}\hat{D}(n)z^n$. The generating function, for moving from j to $j+1$ in one time unit is $p_j z$. The generating functions $T_{i,N}(z)$ satisfy the following recursion relation:

$$T_{i,N}(z) = \frac{p_i z}{1-Q_{i+1,N}(z)}T_{i+1,N}(z), \quad (1)$$

$$0 \le i \le N-1.$$

This is because in the walk (subject to the restrictions in the definition of $T_{i,j}$), one has to move from i to $i+1$ (explaining $p_i z$), then one can move from $i+1$ back to i without reaching N [corresponding to $(1-Q_{i+1,N})^{-1}$], and finally one has to move from $i+1$ to N without returning to $i+1$. Noting that $T_{N-1,N}(z) = p_{N-1}z$, one obtains from Eq. (1),

$$T_{0,N}(z) = p_{N-1}z\prod_{j=0}^{N-2}\frac{p_j z}{1-Q_{i+1,N}(z)}. \quad (2)$$

Arguments similar to those leading to Eq. (1) yield

$$G_{0,N}(z) = [1-q_0 z - Q_{0,N}(z)]^{-1}T_{0,N}(z). \quad (3)$$

A recursion relation for $Q_{i,N}$ is similarly derived:

$$Q_{i,N}(z) = \frac{p_i q_{i+1} z^2}{1-Q_{i+1,N}(z)}, \quad 0 \le i < N. \quad (4)$$

Obviously $Q_{N-1,N}(z) = 0$. In what follows, we denote

$A(z=1)=A$ for any function $A(z)$ and $A' \equiv (dA/dz)_{z=1}$. Obviously, $G_{0,N}=1$ (since the probability to ever reach N is unity). The MFT, $\bar{t}$, is $(d\ln G/dz)_{z=1}$, or $G'_{0,N}$. Hence, from Eq. (3) [and with use of Eq. (4) to simplify the result],

$$\bar{t}=N+\frac{q_0+Q'_{0,N}}{1-q_0-Q_{0,N}}+\sum_{i=0}^{N-2}\frac{Q_{i,N}Q'_{i+1,N}}{p_i q_{i+1}}. \quad (5)$$

It follows from Eq. (4) that

$$Q'_{i,N}=2\sum_{k=i}^{N-2}\left[\prod_{j=1}^{k-1}\frac{Q^2_{j,N}}{p_j q_{j+1}}\right]Q_{k,N}. \quad (6)$$

When $k-1<i$, the product in Eq. (6) is to be replaced by 1. The same convention holds below as well. As a last preparatory step, we wish to find $Q_{0,N}$. Let K_j be the probability[7] for a walker starting at j ($0 \le j \le N$) to reach 0 before reaching N. Obviously, $K_0=1$ and $K_N=0$. The quantities K_j satisfy the master equation $K_j=p_jK_{j+1}+q_{j+1}K_{j-1}$. Solving this equation with the above boundary conditions, we find

$$K_1=1-\left[\sum_{r=1}^{N}\prod_{j=1}^{r-1}\frac{q_j}{p_j}\right]^{-1}. \quad (7)$$

Clearly, $Q_{0,N}=p_0K_1$. Since $p_i=Q_{i,N}+T_{i,N}$ (after leaving i to the right, one either reaches first N or first returns to i). We have $Q_{i,N} \le p_i$. This inequality can be used to obtain further bounds. If we use Eq. (6) and the above inequality,

$$\frac{Q_{i,N}Q'_{i+1,N}}{p_i q_{i+1}} \le 2\sum_{m=i+1}^{N-2}\left[\prod_{r=i+1}^{m}\frac{p_r}{q_r}\right]. \quad (8)$$

Similarly, using Eqs. (6) and (7) one obtains

$$\frac{q_0+Q'_{0,N}}{1-q_0-Q_{0,N}} \le \frac{q_0}{p_0}+2\sum_{m=1}^{N}\sum_{r=1}^{N}\left[\prod_{n=1}^{m-1}\prod_{j=1}^{r-1}\frac{p_n}{q_n}\frac{q_j}{p_j}\right].$$

Performing the double summation one finds

$$\frac{q_0+Q'_{0,N}}{1-q_0-Q_{0,N}} \le \frac{q_0}{p_0}+2N+2\sum_{m=2}^{N}\sum_{r=1}^{m-1}\left[\prod_{k=r}^{m-1}\frac{p_k}{q_k}+\prod_{k=r}^{m-1}\frac{q_k}{p_k}\right]. \quad (9)$$

Substituting Eqs. (8) and (9) in Eq. (5) one obtains an upper bound for $\bar{t}$. Define now $\pi(\alpha,\beta)=\prod_{m=\alpha}^{\beta}q_m/p_m$, then the following inequality holds:

$$\bar{t}<4N+4\sum[\pi(\alpha,\beta)+\pi^{-1}(\alpha,\beta)], \quad (10)$$

where the sum is over all intervals $[\alpha,\beta] \in [0,N]$. For a given realization, let M denote the maximal value of $\pi(\alpha,\beta)$ and $\pi^{-1}(\alpha,\beta)$. The number of segments $[\alpha,\beta]$ ($\beta \ge \alpha$) is $(N+1)(N+2)/2$, i.e., smaller than N^2. Hence, $\bar{t}<4N+4N^2M$. Define $\xi_i=\log(p_i/q_i)$. The quantity $\log\pi(\alpha,\beta)=\sum_{i=\alpha}^{\beta}\xi_i$ is a displacement in a one-dimensional random walk, whose statistics is that of the ξ_i's. The maximal value of $\log\pi$ in a segment is smaller than twice the span[16] of the walk (in N steps). The probability distribution for such spans is known.[16] Using it, we find $\text{Prob}[\max(\log\pi)>N^{1/2+\epsilon}]<c/N^{\epsilon}$ for every $\epsilon>0$; c is an $O(1)$ constant. Define $\eta_i=\log(q_i/p_i)$. A similar calculation yields

$$\text{Prob}[\max(\log\pi^{-1})>N^{1/2+\epsilon}]<c_1/N^{\epsilon}.$$

Altogether, we find that for every $\epsilon>0$

$$\text{Prob}\{\max[\pi(\alpha,\beta)+\pi^{-1}(\alpha,\beta)]>\exp(N^{1/2+\epsilon})\}<k/N^{\epsilon},$$

where k is an $O(1)$ constant. Hence,

$$\text{Prob}[\bar{t}>\exp(hN^{1/2+\epsilon})]<k/N^{\epsilon},$$

where h is an $O(1)$ constant. We have thus shown that $\bar{t}<\exp(hN^{1/2+\epsilon})$ with a probability larger than $1-k/N^{\epsilon}$. Next, we compute a lower bound for $\bar{t}$. It is easy to see from Eq. (5) that

$$\bar{t}>\frac{q_0}{1-q_0-Q_{0,N}}. \quad (11)$$

Using $G_{0,N}=1$ and Eq. (3), we see that the right-hand side of Eq. (11) can be replaced by $q_0/T_{0,N}$. It now follows from Eqs. (3) and (4) that

$$\frac{1}{1-q_0-Q_{0,N}}=\frac{1}{T_{0,N}}=\frac{\prod_{j=0}^{N-2}p_jq_{j+1}}{\prod_{k=0}^{N-1}p_k\prod_{m=0}^{N-2}Q_{m,N}}. \quad (12)$$

Since $Q_{m,N}<p_m$, as shown before, we find, using Eqs. (11) and (12)

$$\bar{t}>\prod_{j=0}^{N-1}\frac{q_j}{p_j}. \quad (13)$$

Consider now a segment of length SN, composed of S subsegments of length N each. The $\{p_\alpha\}$ ($\alpha=0,\ldots,SN$) are chosen from the same distribution as before. Define t_k to be the MFT to go from $j=kN+1$ to $(k+1)N$. Obviously, $\bar{t} \ge \sum_{k=0}^{S-1}t_k$. Using Eq. (13) in each subsegment, we find

$$\bar{t}>\sum_{k=0}^{S-1}\prod_{j=kN+1}^{(k+1)N}\frac{q_j}{p_j}. \quad (14)$$

The S products in Eq. (14) can be regarded as S independent random variables. The probability that at least one of the products is larger than $\exp[(SN)^{1/2-\epsilon}]$ is computed as follows. Let $\zeta_k=\sum_{j=kN+1}^{(k+1)N}\log(q_j/p_j)$. The variables ζ_k have zero mean and a variance of $N\sigma^2$. Their distribution approaches a normal distribution, as $N \to \infty$, by the central limit theorem. Hence

$$\text{Prob}(\zeta_k<x) \simeq \tfrac{1}{2}\{1+\text{erf}[x/(2N)^{1/2}\sigma]\}.$$

Let Pt be the probability that at least one of the ζ_k

exceeds $\sqrt{2}(SN)^{1/2-\epsilon}\sigma$. It follows that

$$Pt \simeq 1-(\tfrac{1}{2})^S[1+\mathrm{erf}(S^{1/2-\epsilon}N^{-\epsilon})^S].$$

Choosing $NS \gg S^{1/2\epsilon}$, we obtain $Pt \simeq 1-(\frac{1}{2})^S$. For sufficiently large S, it follows that $Pt \rightarrow 1$. Since ϵ can be chosen to be arbitrarily small in the above derivation, it follows[16] that $\bar{t} \propto e^{A\sqrt{N}}$ with a probability approaching unity as $N \rightarrow \infty$ [A is an $O(1)$ number]. The main contribution to the MFT is thus due to a subsegment which provides the largest bias motion to the desired endpoint.

Next we turn to computing the realization averaged MFT, $\langle t \rangle$. It follows from Eq. (13) that

$$\langle t \rangle \geq \langle q/p \rangle^N = \exp(N \log\langle q/p \rangle), \tag{15}$$

where $\langle q/p \rangle$ is the realization average of q_j/p_j. It is easy to show that $\langle q/p \rangle > 1$ [based on $\langle \log(q/p) \rangle = 0$]. A second inequality is obtained by our noting that

$$\pi(\alpha,\beta) \leq \prod_{j=\alpha}^{\beta}\left(\frac{q_j}{p_j}+\frac{p_j}{q_j}\right) \leq \prod_{j=0}^{N-1}\left(\frac{q_j}{p_j}+\frac{p_j}{q_j}\right).$$

A similar inequality holds for π^{-1}. Hence, from Eq. (10), we obtain for *every* realization

$$\bar{t} \leq 4N + 4N^2 \prod_{j=0}^{N-1}\left(\frac{q_j}{p_j}+\frac{p_j}{q_j}\right).$$

Averaging over realizations, we find $\langle t \rangle < e^{\lambda N}$, where $\lambda = [\langle (q/p)+(p/q) \rangle] > 1$. Both above bounds for $\langle t \rangle$ can be further improved, but there is no need if all we want to show is that $\log\langle t \rangle \propto N$ for large N. This result should be contrasted to the typical behavior $\log\bar{t} \propto \sqrt{N}$. The reason for this difference is in the fact that very long mean first-passage times have low probability but contribute significantly to the average over-all realizations. Interestingly, the use of the replica method[17] to analyze[8] the Sinai problem which involves averaging over all realizations leads to the typical result and not the averaged one.

A simple model, which is a toy version of the above result, will serve to understand the success of the replica method: Let X be a random variable assuming two values: $e^{\alpha\sqrt{N}}$, with probability $(1-e^{-N})$ and $e^{\beta N}$ with probability e^{-N}, where $N \gg 1$. Obviously, the typical value of X is $e^{\alpha\sqrt{N}}$, whereas the average value, when $\beta > 1$, is approximately $e^{(\beta-1)N}$, which is atypical. The average of X^n, for small enough positive n, is easily seen to be approximately $e^{n\alpha\sqrt{N}}$, thus corresponding to the typical result. Similarly, the average of $\log X$ is dominated by the typical result.

In the realm of statistical mechanics, consider a problem in a random system having N dynamical degrees of freedom (e.g., N spins with random couplings). Let $p(\sigma)$ be the probability that the free energy, F, per degree of freedom takes on the value σ. Let $Z = e^{-FN}$ be the corresponding partition function (the temperature factor is absorbed in F). The average of any power, n, of Z is $\langle Z^n \rangle = \int p(\sigma) e^{-\sigma N n} d\sigma$.

For large values of N, the above average is a result of the competition between the "tendency" of $e^{-\sigma N n}$ to prefer large negative values of σ and the fact that $p(\sigma)$ may be vanishing for too large such values. It is easy to construct models in which $\langle Z^n \rangle$ is dominated by atypical values of σ. When N is kept fixed and n is allowed to go to zero, $\langle Z^n \rangle \simeq 1 - Nn\int \sigma p(\sigma) d\sigma$, thus obtaining a result pertaining to the typical values of σ. We conclude that averaging of a free energy, or using the replica method amplifies the contribution of the typical configurations, which are not necessarily those having the lowest value of the free energy (as a function of the random parameter). Further discussion of this point is left for a future publication.

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