Measure of *CP* Nonconservation and Its Consequence on the Structure of the Kobayashi-Maskawa Matrix

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By use of unitarity of the Kobayashi-Maskawa matrix, the rephasing-invariant measure $\delta_{\rm KM}$ of *CP* nonconservation is evaluated in terms of the moduli squared of four matrix elements, which are taken as the four independent parameters. The striking feature of such a formulation is that the positivity of $\delta_{\rm KM}^2$ gives both upper and lower bounds on the matrix elements $|V_{cd}|$, $|V_{cs}|$, $|V_{td}|$, and $|V_{ts}|$.

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Within the minimal standard model with three generations, all *CP*-nonconserving phenomena arise from the presence of a physical complex phase in the Kobayashi-Maskawa (KM) matrix[1] that appears as a result of the diagonalization of the quark mass matrices. However, the diagonalization of the quark mass matrices leaves a certain arbitrariness in the definition of the fields corresponding to the physical states, since they can be rephased. In the language of the quark mixing matrix which is described by a 3×3 unitary matrix, this corresponds to the freedom to multiply any row or column by an arbitrary phase factor. This ambiguity has been exploited by many authors[2] to choose the most convenient parametrization of the KM matrix suitable for phenomenology. Since the physical observables are independent of the quark phase convention, the notion of rephasing invariants (parametrization independent) was introduced in the three-generation[3] case and generalized to four[4] and N generations[5] to dispel this ambiguity in the formulation of *CP*-nonconservation parameters. The rephasing invariants, combined with a parametrization of the KM mixing matrix with a set of parameters that would have direct relations to measurable quantities, have stimulated interest in this problem. In fact, it has been recognized[6] that a parametrization of the KM matrix in terms of measurable quantities is very efficient and useful to phenomenology. For instance, a step along this direction was proposed recently by Bjorken and Dunietz.[7]

In this Letter, I suggest a general parametrization of the KM matrix in which $\delta_{\rm KM}$, the measure of *CP* nonconservation in the three-generations model, is expressed in terms of the moduli squared of four matrix elements, which I take for definiteness as $|V_{ud}|^2$, $|V_{cd}|^2$, $|V_{ub}|^2$, and $|V_{cb}|^2$. These, then, suffice to determine the moduli squared of the remaining five elements of the mixing matrix. This parametrization is automatically rephasing invariant since it is expressed in terms of the modulus squared of the matrix elements. To achieve such parametrization, the determination of the quartic and higher-order rephasing invariants in terms of the quadratic ones is necessary. These quantities, as recently emphasized by Nieves and Pal,[5] enter in all charged-current processes involving Dirac fermions including the *CP*-conserving ones. In addition, they showed that all higher-order rephasing invariants (more than four) can be expressed in terms of the quadratic and quartic ones. Then the task is to evaluate the quartic rephasing invariants in terms of the quadratic ones in order to accomplish the proposed parametrization. Following this parametrization, I propose an approach for determining the KM mixing matrix from the analysis of $\delta_{\rm KM}$. It is based on the positivity of $\delta_{\rm KM}^2$ to determine the allowed range of $|V_{cd}|$, $|V_{cs}|$, $|V_{td}|$, and $|V_{ts}|$ from the values of the elements $|V_{ud}|$, $|V_{us}|$ and the existing range for the elements $|V_{ub}|$ and $|V_{cb}|$. One may wonder at this stage why I do not use the existing values of the matrix elements to determine $\delta_{\rm KM}$. Clearly, such analysis is not feasible at this moment because of the large uncertainties in the determination of the matrix elements $|V_{i\alpha}|$.

I begin by defining the *CP* nontrivial rephasing invariant formed by the product of four of the matrix elements,

$$T_{i\alpha j\beta}=V_{i\alpha}V_{j\beta}V_{i\beta}^{*}V_{j\alpha}^{*}. \tag{1}$$

As is well known, CP nonconservation is always discussed in terms of the imaginary and real parts of the above quantity. The remarkable feature of the three-generation model, as noticed by many authors,[3] is that all the nine imaginary parts of $T_{i\alpha j\beta}$ contribute equally to *CP* nonconservation, namely,

$$\mathrm{Im}(T_{i\alpha j\beta})=\pm\delta_{\rm KM}, \tag{2}$$

where $\delta_{\rm KM}=c_1c_2c_3s_1^2s_2s_3\sin\delta$ in the KM parametrization is the measure of *CP* nonconservation in the standard model. Since all $\mathrm{Im}(T_{i\alpha j\beta})$ are equal up to a sign, expressing one of them in terms of $|V_{i\alpha}|^2$ is sufficient for this study. For example let us take

$$T_{ubcd}=V_{ub}V_{cd}V_{ud}^{*}V_{cb}^{*}. \tag{3}$$

From the orthogonality of the first and second rows of the mixing matrix V, we have

$$V_{cd}V_{ud}^{*}+V_{cs}V_{us}^{*}+V_{cb}V_{ub}^{*}=0. \tag{4}$$

Multiplying Eq. (4) by $V_{ub}V_{cb}^*$, we obtain

$$V_{ub}V_{cd}V_{ud}^*V_{cb}^* + V_{ub}V_{cs}V_{cb}^*V_{us}^* = -|V_{ub}|^2|V_{cb}|^2. \quad (5)$$

Taking the imaginary part of both sides of Eq. (5), one finds

$$\mathrm{Im}(V_{ub}V_{cd}V_{ud}^*V_{cb}^*) = -\mathrm{Im}(V_{ub}V_{cs}V_{us}^*V_{cb}^*). \quad (6)$$

This suggests that we write

$$V_{ub}V_{cd}V_{ud}^*V_{cb}^* = x + i\delta_{\mathrm{KM}} \quad (7a)$$

and

$$V_{ub}V_{cs}V_{us}^*V_{cb}^* = y - i\delta_{\mathrm{KM}}, \quad (7b)$$

which translates, by use of Eqs. (5) and (7), to

$$x + y = -|V_{ub}|^2|V_{cb}|^2. \quad (8)$$

The real part of $V_{ub}V_{cd}V_{ud}^*V_{cb}^*$ is obtained by our taking the modulus squared of the unitarity relation $V_{cd}V_{ud}^* + V_{cb}V_{ub}^* = -V_{cs}V_{us}^*$:

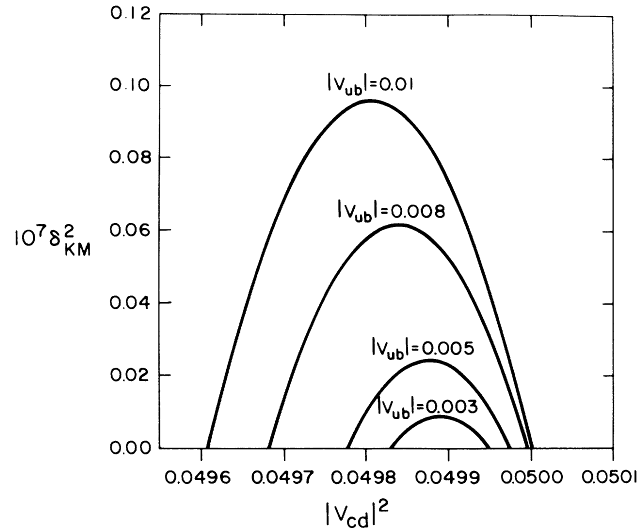

FIG. 1. The matrix element $|V_{cd}|^2$ vs δ^2_{KM} showing the allowed range for $|V_{cd}|$ for each of the four cases, $|V_{ub}| = 0.003$, $|V_{ub}| = 0.005$, $|V_{ub}| = 0.008$, and $|V_{ub}| = 0.01$, in which the values of V_{ud} and V_{cb} are fixed to be $|V_{ud}| = 0.9747$, and $|V_{cb}| = 0.045$.

$$|V_{cd}|^2|V_{ud}|^2 + |V_{cb}|^2|V_{ub}|^2 + 2\mathrm{Re}(V_{ub}V_{cd}V_{ud}^*V_{cb}^*) = |V_{cs}|^2|V_{us}|^2, \quad (9)$$

which leads to

$$\mathrm{Re}(V_{ub}V_{cd}V_{ud}^*V_{cb}^*) = x = \tfrac{1}{2}(|V_{cs}|^2|V_{us}|^2 - |V_{cd}|^2|V_{ud}|^2 - |V_{cb}|^2|V_{ub}|^2). \quad (10)$$

Hence from Eqs. (7) and (10), δ_{KM} can be exactly expressed in terms of the moduli squared of the matrix elements as

$$\begin{aligned}\delta^2_{\mathrm{KM}} &= |V_{ub}|^2|V_{cb}|^2|V_{ud}|^2|V_{cd}|^2 - x^2 \\ &= |V_{ub}|^2|V_{cb}|^2|V_{ud}|^2|V_{cd}|^2 - (|V_{cs}|^2|V_{us}|^2 - |V_{cd}|^2|V_{ud}|^2 - |V_{ub}|^2|V_{cb}|^2)^2/4 \end{aligned} \quad (11)$$

which, in terms of the four basis parameters, becomes

$$\begin{aligned}\delta^2_{\mathrm{KM}} = {} & |V_{ub}|^2|V_{cb}|^2|V_{ud}|^2|V_{cd}|^2 \\ & -(1-|V_{ud}|^2-|V_{cd}|^2-|V_{cb}|^2-|V_{ub}|^2+|V_{ud}|^2|V_{cb}|^2+|V_{ub}|^2|V_{cd}|^2)^2/4. \end{aligned} \quad (12)$$

The above relation is valid only in the three-generation case, and thus when accurate data on the $|V_{ia}|$ become available will put the KM model under severe test. Also, it contains an ambiguity in the sign of δ_{KM}. However, this is no problem in the present case since I am using δ^2_{KM} in a study of its implications on the structure of the KM matrix rather than δ_{KM}. The result on δ^2_{KM} [see Eq. (12)] in terms of the $|V_{ia}|^2$ enables us to determine the upper and lower bounds of the matrix elements $|V_{cd}|$, $|V_{cs}|$, $|V_{td}|$, and $|V_{ts}|$. In this case, for example, to determine $|V_{cd}|^2$ from the positivity of δ^2_{KM}, we fix $|V_{ud}|$ and vary $|V_{ub}|$ and $|V_{cb}|$ in their allowed range. The allowed ranges for V_{ub} and V_{cb}, which is extracted from experiment, depend on the assumed model for strong interactions.[8] I have used the whole range for V_{ub} and V_{cb} to reduce the study to one analysis which includes all these models.[8] Following the same procedure, one can determine the ranges for $|V_{cs}|$, $|V_{td}|$, and $|V_{ts}|$ by choosing the appropriate variables. To illustrate this point, I have plotted, in Fig. 1, the allowed region for $|V_{cd}|$ for the case $|V_{ud}| = 0.9747$ and $|V_{cb}| = 0.045$ with $|V_{ub}|$ taking several values. The obtained KM matrix in absolute values is summarized as

$$|V_{\mathrm{KM}}| = \begin{pmatrix} 0.9742 \text{ to } 0.9752 & 0.220 \text{ to } 0.222 & 0.003 \text{ to } 0.010 \\ 0.2223 \text{ to } 0.2236 & 0.9731 \text{ to } 0.9747 & 0.035 \text{ to } 0.065 \\ 0.0026 \text{ to } 0.0240 & 0.032 \text{ to } 0.066 & 0.9980 \text{ to } 0.9994 \end{pmatrix}. \quad (13)$$

We observe that the magnitude of the $|V_{i\alpha}|$ displayed above suggests that the KM matrix is fairly asymmetric.[9] Further information on the $|V_{i\alpha}|$ can be obtained from CP nonconservation in the neutral kaon and B-meson systems if one believes that $\delta_{\rm KM}$ is the only source of CP nonconservation in the standard model.[9] To do such an analysis we need to express the KM structure of these CP-nonconservation parameters in terms of the moduli squared of the matrix elements of V. This study, which might serve as a possible way to understand the structure of the mixing matrix, is under way.[10] I also attempt, in the same study, to generalize such parametrization to the four-generation case and its consequences on the B-meson decay asymmetry and B-$\bar{B}$ mixing.

I now turn to the implications of such formulation on the structure of the KM mixing matrix. To illustrate, I study the case where V_{ub} is very small or zero. As is well known, if $V_{ub}=0$, there is no CP nonconservation. Setting $V_{ub}=0$ in the relation given by Eq. (12), one obtains $\delta_{\rm KM}^2=0$ since $|V_{ud}|^2|V_{cd}|^2$ is equal to $|V_{cs}|^2|V_{us}|^2$ in that limit [see Eqs. (9) and (11)] and in terms of the chosen variables, it leads to

$$1-|V_{ud}|^2-|V_{cd}|^2-|V_{cb}|^2+|V_{ud}|^2|V_{cb}|^2 = 0. \qquad (14)$$

In fact, what is nice about this parametrization is that none of the matrix elements can be zero if CP is not conserved in nature *à la* Kobayashi-Maskawa. It is very clear from Eqs. (9) and (11) that if any of the matrix elements is zero $\delta_{\rm KM}$ vanishes and the mixing matrix can be made real. Furthermore, by looking at the relation $|V_{ud}|^2|V_{cd}|^2=|V_{us}|^2|V_{cs}|^2$ which is obtainable from Eq. (9) in the limit V_{ub} goes to zero, one sees that if one of the nine two-by-two submatrices is unitary, the $T_{i\alpha j\beta}$ are real and there is no CP nonconservation. It is therefore crucial to determine the matrix element V_{ub} to understand further the structure of the KM matrix.

In conclusion, I have emphasized the importance of the rephasing invariants, in particular $\delta_{\rm KM}$, and the consequences they have on the structure of the KM mixing matrix. I have shown that $\delta_{\rm KM}$ can be written as a function of the moduli squared of the KM matrix elements, therefore enabling us to use directly the experimental data on these elements when available. I have also shown in a simple way that the standard three-generation model does not exhibit CP nonconservation if any of the KM matrix elements is zero and/or if any of the nine submatrices of the KM matrix is unitary. The positivity of $\delta_{\rm KM}^2$ gives both upper and lower bounds on the matrix elements $|V_{cd}|$, $|V_{cs}|$, $|V_{td}|$, and $|V_{ts}|$. The determination of V_{ub} is urgently needed for either refinement or testing of the KM matrix. These conclusions are valid only in the three-generation standard model. This may serve as a good test for the KM model, when improved theoretical and experimental results on the CP-nonconservation quantities in the neutral K and B meson systems and accurate measurements of the $|V_{i\alpha}|$ matrix elements become available.

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After completing this work, I received two preprints, one from Jarlskog[11] and the second from Branco and Lavoura,[12] where they parametrize the mixing matrix in terms of the moduli of the matrix elements. Their result on the expression of $\delta_{\rm KM}$ in terms of $|V_{i\alpha}|$ is in agreement with mine.

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