# Nonlinear Focusing of Coupled Waves

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The collinear propagation of an arbitrary number of finite-amplitude waves is modeled by a system of coupled nonlinear Schrödinger equations. This model incorporates the effects of a coordinate-dependent external potential and coordinate-independent nonlinearities. A virial theorem is derived, which governs the nonlinear focusing of these coupled waves. For two light waves in a beat-wave accelerator, the coupling to each other and to the resonantly generated Langmuir wave has a significant effect on the focusing properties of the waves.

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In the plasma beat-wave accelerator,[1] a large-amplitude Langmuir wave is generated by the beating of two collinear lasers whose frequencies differ by approximately the plasma frequency. The longitudinal electric field of this Langmuir wave can then be used to accelerate particles. Since any reduction in laser amplitude results in a corresponding reduction in the accelerating field, it is important that the laser beams remain focused as they propagate through the plasma. For parameters typical of a proposed beat-wave accelerator, the Rayleigh diffraction length of the incident laser beams is much less than the particle-acceleration length and a mechanism for nonlinear focusing is required.

Such a mechanism is the relativistic correction to the mass of electrons oscillating in the electric fields of the incident light waves. This alters the nonlinear index of refraction in such a way that energy accumulates around local maxima in the wave amplitudes. Previous analyses of this process[2] have dealt with the steady-state focusing of a single light wave. Although these analyses have produced valuable insight into the process of relativistic self-focusing, they are somewhat incomplete. Since the electron "quiver" velocity depends on both light-wave amplitudes, the nonlinear focusing of *either* wave depends on the amplitude of *both* waves. Moreover, the incident pulse lengths are so short that the steady-state assumption is invalid. In this Letter, the time-dependent focusing of nonlinearly coupled light waves is studied and the effects of a resonantly generated Langmuir wave are discussed.

Consider a system of light waves whose group velocities are parallel and of approximately equal magnitude. In a Lorentz frame moving with the average group velocity, the spatiotemporal evolution of the wave amplitudes is governed by the Lagrangian density

$$L=\tfrac{1}{2}i(A_\alpha^*\,\partial_t A_\alpha - A_\alpha\partial_t A_\alpha^*)-\tfrac{1}{2}(\nabla A_\alpha^*\cdot\nabla A_\alpha)+Q(t,\mathbf{x},|A_\alpha|^2), \quad (1)$$

where the canonical variables are the wave amplitudes $A_\alpha$ and $A_\alpha^*$ and, with the exception of Eq. (2), repeated wave subscripts imply summation. For the beat-wave application which motivated this paper, $t$ and $\mathbf{x}$ are both measured in units of the collisionless skin depth $c\omega_e^{-1}$, the $A_\alpha$'s are the electron quiver velocities divided by the speed of light, and the potential $Q$ has a specific functional form. However, a more general potential is retained in the analysis: The only restriction on the potential is that it can be written as the sum of a real external potential $Q_e$, which depends on $t$ and $\mathbf{x}$, and is at most linear in $|A_\alpha|^2$, and an internal potential $Q_i$, which is a real function of $|A_\alpha|^2$ alone. Application of the Euler-Lagrange equations to the Lagrangian (1) generates the coupled nonlinear Schrödinger equations

$$[i\,\partial_t+\tfrac{1}{2}\nabla^2+Q_\alpha(t,\mathbf{x},|A_\beta|^2)]A_\alpha=0\,, \quad (2)$$

where $Q_\alpha$ denotes $\partial Q/\partial|A_\alpha|^2$. Since the potential functions $Q_\alpha$ are real, no energy is exchanged among the waves and each wave action

$$N_\alpha=\int|A_\alpha|^2 d^Dx$$

is conserved. Additional conservation laws can be deduced from the stress-energy tensor[3]

$$T_\nu^\mu=A_{\alpha,\nu}\left(\frac{\partial L}{A_{\alpha,\mu}}\right)+A_{\alpha,\nu}^*\left(\frac{\partial L}{A_{\alpha,\mu}^*}\right)-g_\nu^\mu L\,, \quad (3)$$

where the subscript $,\mu$ denotes $\partial/\partial x^\mu$, and the divergence conditions

$$\partial_\mu T_\nu^\mu=-\partial_\nu L\,. \quad (4)$$

It follows from the temporal components of Eqs. (3) and (4) that

$$H=\int[\tfrac{1}{2}(\nabla A_\alpha^*\cdot\nabla A_\alpha)-Q(t,\mathbf{x},|A_\alpha|^2)]d^Dx$$

is the total wave energy, which evolves according to

$$\frac{d}{dt}H=-\int\partial_t Q_e\,d^Dx\,. \quad (5)$$

It follows from the spatial components of Eqs. (3) and (4) that

$$\mathbf{P}=\frac{1}{2i}\int(A_\alpha^*\nabla A_\alpha-A_\alpha\nabla A_\alpha^*)d^Dx$$

is the total wave momentum, which evolves according to

$$\frac{d}{dt}\mathbf{P}=\int \nabla Q_e\, d^D x\,. \qquad (6)$$

Because of the complexity of Eq. (2), attention will be focused on the temporal evolution of certain spatially averaged properties of the wave envelopes. In this "moment" or "virial" approach,[4] the average value $\langle \dots \rangle$ of a physical quantity is defined as the integral $\int A_\alpha^* A_\alpha d^D x$ divided by the total wave action $N$. For example, the velocity $d_t\langle x\rangle$ of the centroid of the wave envelopes is equal to $\mathbf{P}/N$. The average width $\langle \delta x^2\rangle^{1/2}$ of the wave envelopes is defined as $\langle(\mathbf{x}-\langle\mathbf{x}\rangle)^2\rangle^{1/2}$. By use of the Schrödinger equations (2) to replace the time derivatives $\partial_t A_\alpha^{(*)}$ with terms involving $\nabla A_\alpha^{(*)}$ and $|A_\beta|^2$, and by performance of the requisite integrations by parts, it is not difficult to show that

$$\frac{d^2}{dt^2}\langle \delta x^2\rangle = 2\left[2\left(\frac{H}{N}\right)-\left(\frac{P}{N}\right)^2-\langle\mathbf{x}\rangle\cdot\frac{d}{dt}\left(\frac{\mathbf{P}}{N}\right)+R\right], \qquad (7)$$

where the remainder term

$$R=\frac{1}{N}\int[2Q-DQ_e+D(Q_i-Q_{i\alpha}|A_\alpha|^2)]d^D x\,. \qquad (8)$$

These virial equations can be generalized to include waves with anomalous dispersion.[5]

For two light waves in a beat-wave accelerator, with associated electron quiver velocities which are only weakly relativistic, the potential takes the form[5]

$$Q_e=O,\quad Q_i=\tfrac{1}{2}\Lambda_{\alpha\beta}|A_\alpha|^2|A_\beta|^2, \qquad (9)$$

where $\Lambda_{11}$ and $\Lambda_{22}$ are equal to $\frac{1}{8}$, $\Lambda_{12}$ and $\Lambda_{21}$ are equal to $\frac{1}{4}$, and the effects of the resonantly generated Langmuir wave have been neglected temporarily. It will henceforth be assumed that the incident light waves are only a few collisionless skin depths wide. This narrow-wave assumption is appropriate for the beat-wave accelerator and has two important consequences: First, the waves are unlikely to filament[5] and a discussion of whole-wave focusing is meaningful; and second, the time scale for transverse focusing[4] (which is proportional to the product of the square of the envelope width and $|A_\alpha|^{-1}$) is much shorter than the time scale for the longitudinal modulational instability[5,6] (which is proportional to $|A_\alpha|^{-2}$). Because of this disparity in time scales, the longitudinal derivatives in the Laplacian can be neglected when we consider the initial development of transverse focusing. Suppose then, that the incident wave envelopes are given by

$$A_\alpha(0,z,r)=\left[\frac{N_\alpha(z)}{\pi\rho^2(z)}\right]^{1/2}\exp\left[-\frac{r^2}{2\rho^2(z)}\right],$$

and that $D$ is equal to 2. It follows from the initial conditions and the Ehrenfest equation (6) that $\mathbf{P}$ and $d_t\mathbf{P}$ are both equal to zero, and, from the choice of dimension and the form of the potential that $R$ is equal to zero. Only the term $H$ contributes to Eq. (7), which reduces to

$$\frac{d^2}{dt^2}\langle\delta r^2\rangle=\frac{2}{N\rho^2}[N-\lambda_{\alpha\beta}N_\alpha N_\beta]\,, \qquad (10)$$

where $\lambda_{\alpha\beta}$ is equal to $\Lambda_{\alpha\beta}/2\pi$ and the parametric dependence of $\langle\delta r^2\rangle$, $N_\alpha$, and $\rho$ on $z$ has been suppressed for simplicity of notation. By virtue of Eq. (5), $H$ is constant. It follows that the envelope width will collapse to zero on a finite time whenever $H$ is negative. This collapse is an artifact of an idealized physical model, but is a useful numerical diagnostic because the onset of nonlinear focusing is independent of the physical mechanisms which limit nonlinear focusing. Notice that waves which do not focus by themselves will often focus because of their collective interaction. If we assume that the critical power is exceeded, Eq. (10) can be integrated to determine the collapse time $t_c$. For wave envelopes with no initial divergence,

$$t_c=\rho^2[N/(\lambda_{\alpha\beta}N_\alpha N_\beta-N)]^{1/2}. \qquad (11)$$

For a single light wave, the critical value of $N(z)$, above which the wave envelope will collapse, is equal to $16\pi$. In physical units, this corresponds to a critical power of $1.7\times10^{10}(\omega_l/\omega_e)^2$ W, in agreement with Refs. 2. However, it should be emphasized that only those portions of the wave for which $N(z)$ is greater than $N_c$ will collapse; the other portions of the wave will disperse. This invalidates the steady-state assumption of Refs. 2, which requires the wave to collapse or disperse as a single entity. For two copropagating light waves, the nonlinear coupling reduces the critical power required in each wave by a factor of 3. It should be emphasized also that Eq. (11) refers to the collapse time which would be observed in the group frame; the plasma-frame collapse time is longer than the group-frame collapse time by a factor of $\omega_l/\omega_e$.

The predictions of virial theory were tested by our solving the relevant two-dimensional Schrödinger equations numerically.[7] For the cases described in the preceding paragraph, the negativity of $H$ was found to be a sufficient condition for the transverse collapse of the wave envelope. The time taken for the development of a singularity in the wave amplitude was of the order of, but less than, the predicted collapse time (11). This is because, in a typical collapse, not all of the wave action is localized near the singularity. After the singularity has occurred, the assumptions made in the derivation of the virial equation (10) break down. No predictions can then be made, based on the initial conditions, regarding the fate of the wave action not localized near the singularity.

Now consider the evolution of two waves whose centroids $\langle\mathbf{x}\rangle_\alpha$ are separated by a finite distance. In Fig. 1, snapshots of $|A_1|^2$ are displayed for several values of

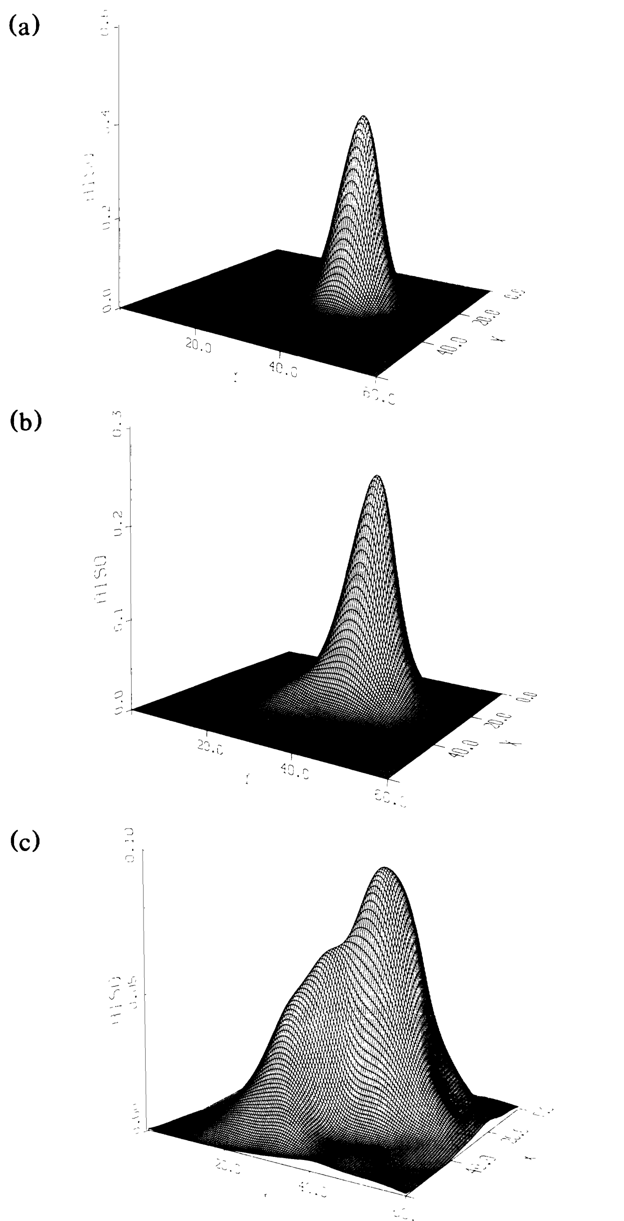

FIG. 1. Snapshots of $|A_1|^2$ for several values of the elapsed time; the corresponding snapshots of $|A_2|^2$ can be inferred by reflection in the center of the simulation box. Each dimensionless wave action is equal to 33 and the initial separation of the wave centroids is equal to $2.2\rho$, where $\rho$ is the Gaussian scale length of the wave envelopes. (a) $t=0$; (b) $t=45$; (c) $t=90$.

the elapsed time; the corresponding snapshots of $|A_2|^2$ can be inferred by reflection in the center of the simulation box. Each wave action is equal to 33 and the initial separation of the wave centroids is equal to $2.2\rho$. Initially, the waves are well separated and there is little interaction between them. Since neither wave has enough action to collapse by itself, both waves begin to disperse. However, as the waves disperse, their spatial extent increases and the interaction between them becomes stronger: Their centroids merge and they subsequently evolve as a single entity. This process is known as entrainment. By the end of the numerical simulation shown in Fig. 1, the width of the wave envelopes exceeds that of the simulation box and further numerical evolution is inappropriate. However, numerical simulations with larger boxes show that the waves continue to disperse.

If the waves had been coincident initially, their combined action would have exceeded the critical value for collapse. One is therefore led to ask how large the initial separation can be without suppressing the tendency of the waves to collapse. By evaluation of the integrals in the virial equation (7) for two waves with partially overlapping Gaussian envelopes, a variant of Eq. (10) is obtained in which the interaction terms are reduced by a factor of $\exp[|\langle \mathbf{x}\rangle_1 - \langle \mathbf{x}\rangle_2|^2/2\rho^2]$. For two light waves of equal intensity, the criterion for entrainment and collapse simplifies to

$$|\langle \mathbf{x}\rangle_1 - \langle \mathbf{x}\rangle_2| \le \rho\{2\ln[2N/(3N_c - N)]\}^{1/2}, \qquad (12)$$

where $N_c$ denotes the critical action for two coincident waves. For two light waves whose action is equal to 25, virial theory predicts a tolerable separation of approximately $1.2\rho$. In Fig. 2, snapshots of $|A_1|^2$ are displayed for several values of time, for an initial separation of $1.1\rho$. Because the waves are not well separated initially, the entrainment process is less dramatic than in Fig. 1. However, the waves *do* collapse. This verifies the sufficiency of condition (12). The robustness of coupled nonlinear focusing with regard to small laser misalignments could be important in experiments.

As the incident light waves propagate through the plasma, the component of the ponderomotive force at their beat frequency drives a Langmuir wave whose amplitude grows smoothly from the front of the light waves to the rear.[8] Energy is then transferred from the incident waves to their sidebands by repeated stimulated Raman scattering.[8] A major deficiency of the Schrödinger-equation model is its failure to account for these processes self-consistently. However, in the spirit of Joshi, Clayton, and Chen,[9] one can estimate the Langmuir-wave amplitude $A_p(\mathbf{x})$ for given incident pulse shapes. The focusing effects of the Langmuir wave can then be modeled by the inclusion of an external potential

$$Q_e = \lambda_{p\alpha}|A_p(\mathbf{x})|^2|A_\alpha|^2$$

in the virial equations (7) and (8). Such an analysis showns that near the front of the light waves, the Langmuir wave is of small amplitude and has little effect on the focusing of the light waves. In contrast, near the rear of the light waves, the Langmuir wave is of large amplitude and significantly enhances the focusing tendency of the light waves. Since the virial equation (10) is valid for an arbitrary number of waves, one can also

(a)

(b)

(c)

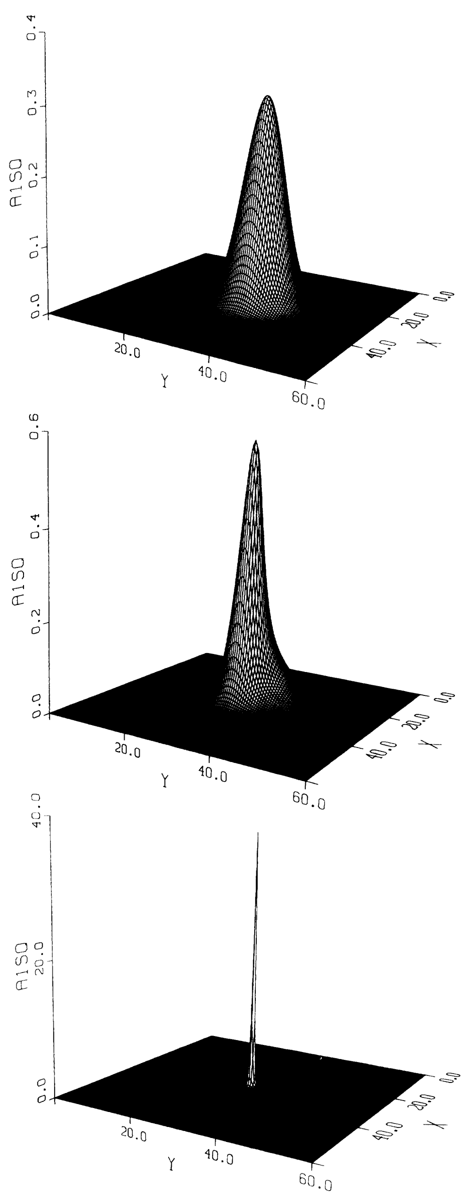


FIG. 2. Snapshots of $|A_1|^2$ for several values of the elapsed time; the corresponding snapshots of $|A_2|^2$ can be inferred by reflection in the center of the simulation box. Each dimensionless wave action is equal to 25 and the initial separation of the wave centroids is equal to $1.1\rho$, where $\rho$ is the Gaussian scale length of the wave envelopes. (a) $t=0$; (b) $t=20$; (c) $t=40$.

model the effects of resonant energy transfer from the incident waves to their sidebands. By use of the potential (9), and the result[5] that the self-nonlinear coefficients $\Lambda_{\alpha\alpha}$ are all equal to $\frac{1}{8}$ and the cross-nonlinear coefficients $\Lambda_{\alpha\beta}$ are all equal to $\frac{1}{4}$, it is easily shown that the tendency of the light waves to focus is not reduced by any assumed redistribution of electromagnetic energy. An independent study of these effects has been made by Gibbon and Bell.[10] It should also be remarked that there are at least three mechanisms which prevent the occurrence of singularities in the wave amplitudes. First, the relativistic nonlinearity is saturable,[2] although its form is unknown for systems of coupled waves. Second, the parametric decay of the constituent waves provides a potentially important energy "sink" and can produce steep transverse gradients in the laser intensities which enhance diffraction.[11] Third, the expulsion of plasma particles from the laser filament due to the pressure of the laser light must cease once vacuum conditions have been attained.[2,12] The collapse criteria (10) and (12) should therefore be regarded as criteria for the *initial occurrence* of relativistic focusing, and the collapse time (11) should be regarded as an *estimate* of the time scale on which higher-order nonlinearities or other physical processes become important.

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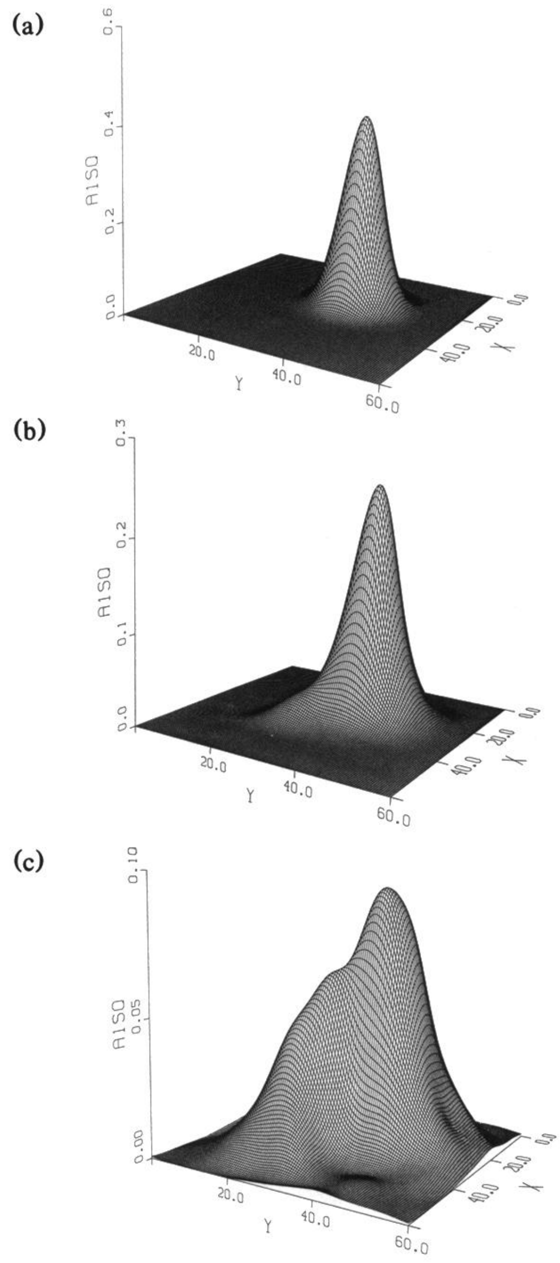


FIG. 1. Snapshots of $|A_1|^2$ for several values of the elapsed time; the corresponding snapshots of $|A_2|^2$ can be inferred by reflection in the center of the simulation box. Each dimensionless wave action is equal to 33 and the initial separation of the wave centroids is equal to $2.2\rho$, where $\rho$ is the Gaussian scale length of the wave envelopes. (a) $t=0$; (b) $t=45$; (c) $t=90$.

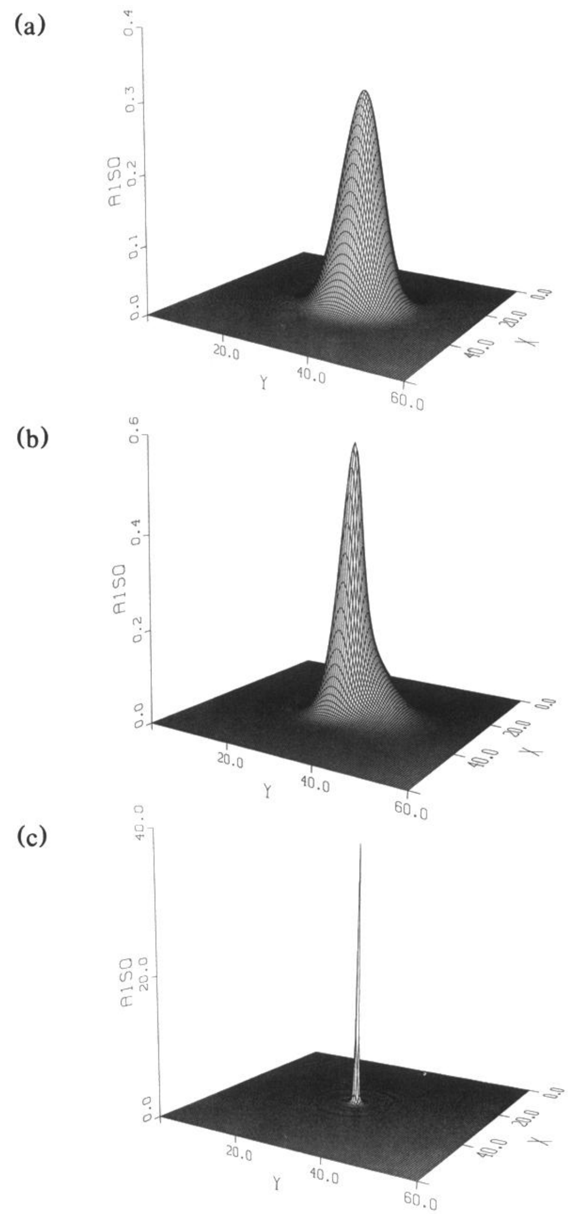


FIG. 2. Snapshots of $|A_1|^2$ for several values of the elapsed time; the corresponding snapshots of $|A_2|^2$ can be inferred by reflection in the center of the simulation box. Each dimensionless wave action is equal to 25 and the initial separation of the wave centroids is equal to $1.1\rho$, where $\rho$ is the Gaussian scale length of the wave envelopes. (a) $t=0$; (b) $t=20$; (c) $t=40$.