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Formal Scattering Theory Approach to S-Matrix Relations in Supersymmetric Quantum Mechanics

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Combining the methods of scattering theory and supersymmetric quantum mechanics we obtain relations between the S matrix and its supersymmetric partner. These relations involve only asymptotic quantities and do not require knowledge of the dynamical details. For example, for coupled channels with no threshold differences the relations involve the asymptotic normalization constant of the bound state removed by supersymmetry.

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Recently, supersymmetry in nonrelativistic quantum mechanics (SSQM)[1] has been used to derive S-matrix relations for potential scattering.[2–4] We extended this method to coupled two-body channels and obtained simple relations between the S matrix and its supersymmetric partner in the case where the channels are distinguished by different threshold energies.[5] In this Letter we obtain general S-matrix relations in SSQM. We develop a more powerful formal scattering theory methodology than we used and hence this work is not a simple extension of our previous work but rather subsumes it. Our new results relate the S matrix to its supersymmetric partner entirely in terms of asymptotic quantities. We derive relations among supersymmetric partner S-matrix elements without recourse to dynamical details.

Since the supersymmetric partner Hamiltonian (as we consider it here) does not have the lowest bound state of the original Hamiltonian, the relationship among the S-matrix elements must remove the corresponding S-matrix pole. In the coupled-channel case that pole is distributed over all the S-matrix elements with residues proportional to the asymptotic normalization of the bound state in the appropriate channel. Therefore, the relation between the S matrix and its supersymmetric partner contains both the bound-state energy and those asymptotic normalization constants.

Consider a matrix Hamiltonian, H, coupling n two-body channels. After partial-wave decomposition the Hamiltonian ($\hbar = 2m = 1$) for the radial Schrödinger equation can be written

$$H = -d^2/dr^2 + V\,, \tag{1}$$

where V contains appropriate centrifugal terms, we suppress the angular momentum label l, and we assume all channel masses to be equal. If the lowest-energy bound state of H, assumed to exist, is ψ_1 with energy eigenvalue $-B$, one can write H in factored form[6–9]

$$H = A^+A^- - B \tag{2}$$

with the usual form for the $A^{\pm}$'s

$$A^{\pm} = \pm d/dr + W\,, \tag{3}$$

where W is an $n \times n$ matrix. The superpotential W can be constructed from ψ_1 and the $n-1$ linearly independent solutions ψ_j ($j = 2, \ldots, n$) of

$$H\psi_j = -B\psi_j \tag{4}$$

that are regular at the origin but grow exponentially at large r. Defining

$$\Psi = (\psi_1, \psi_2, \ldots, \psi_n) \tag{5}$$

as an $n \times n$ matrix with rows made from the column vectors ψ_j we can take

$$W = \Psi'\Psi^{-1} \tag{6}$$

(prime means differentiation with respect to r) and we will obtain the Hamiltonian H from Eqs. (3) and (6). It can be shown[10] that W is Hermitian although this may not be obvious from Eq. (6).

The supersymmetric partner Hamiltonian

$$\tilde{H}=A^{-}A^{+}-B \tag{7}$$

has the same spectrum as H except for the lowest bound state and the unnormalized solutions of $\tilde{H}$ for energy E, $\tilde{\psi}$, are given by

$$\tilde{\psi}_E=A^{-}\psi_E\,. \tag{8}$$

Normalizing $\tilde{\psi}$ to the proper asymptotic state is an important aspect of obtaining the S-matrix relations. We want to relate the S matrix of H, S, to that of $\tilde{H}$, $\tilde{S}$, in a given partial wave in the framework of formal scattering theory.[11] The scattering states of H are given in terms of the Möller wave operators, $\Omega_{\pm}$, by

$$\psi^{\pm}=\Omega_{\pm}\varphi_{\text{in/out}}\,, \tag{9}$$

where the φ_{in} and the φ_{out} are free states with the incoming or outgoing parts properly normalized or, in the language of Newton,[11] "controlled." In terms of the wave operators, the scattering operator is given by

$$S=\Omega_{-}^{\dagger}\Omega_{+} \tag{10}$$

and the radial S-matrix elements in our partial wave, l, by

$$S(k)=\langle\varphi_{\text{out},k}\,|\,\Omega_{-}^{\dagger}\Omega_{+}\,|\,\varphi_{\text{in},k}\rangle\,, \tag{11}$$

where k is related to the scattering energy by $E=k^2$. We require that $\tilde{\psi}^{\pm}$ the corresponding scattering solutions at energy E of $\tilde{H}$ be normalized so that asymptotically they have the appropriate incoming or outgoing spherical wave. For the normalized $\tilde{\psi}^{\pm}$ we write

$$\tilde{\psi}^{\pm}=A^{-}\Omega_{\pm}(A_{\infty,\pm}^{-})^{-1}\varphi_{\text{in-out},k}\,, \tag{12}$$

where $A_{\infty,\pm}^{\pm}$ is given by Eq. (3) with W replaced by its asymptotic form, and the plus-minus sign means that it acts only as the in-out part of the free state, the part that must be recontrolled asymptotically to undo the effect of the A^{-}. The asymptotic superpotential W_∞ is an $n\times n$ constant (independent of r) matrix with the property that $W_\infty^2=B$.

Equation (12) defines the supersymmetric wave operators as

$$\tilde{\Omega}_{\pm}=A^{-}\Omega_{\pm}(A_{\infty,\pm}^{-})^{-1} \tag{13}$$

and the partner scattering operator as

$$\tilde{S}=\tilde{\Omega}_0^{\dagger}\tilde{\Omega}_{+}=(A_{\infty,-}^{+})^{-1}\Omega_{-}^{\dagger}A^{+}A^{-}\Omega_{+}(A_{\infty,+}^{-})^{-1}\,. \tag{14}$$

To obtain this last result we used $(A^{-})^{\dagger}=A^{+}$. From Eq. (2) we have

$$\begin{aligned}\tilde{S}&=(A_{\infty,-}^{+})^{-1}\Omega_{-}^{\dagger}(H+B)\Omega_{+}(A_{\infty,+}^{-})^{-1}\\&=(A_{\infty,-}^{+})^{-1}(H_0+B)S(A_{\infty,+}^{-})^{-1}\,,\end{aligned} \tag{15}$$

where in going from the first to the second line we used the intertwining property,

$$H\Omega_{\pm}=\Omega_{\pm}H_0 \tag{16}$$

and where H_0 is the asymptotic form of H. Now taking matrix elements of $\tilde{S}$ between free states [the appropriate in or out eigenstates of H_0 as in (11)] and using $k^2+(W_\infty)^2=k^2+B$, $B=K^2$ we obtain a set of equivalent forms

$$\tilde{S}(k)=(-ik+W_\infty)S(k)(ik+W_\infty)^{-1} \tag{17a}$$

$$=(ik+W_\infty)^{-1}S(k)(-ik+W_\infty) \tag{17b}$$

$$=(ik+W_\infty)^{-1}(k^2+B)S(k)(ik+W_\infty)^{-1} \tag{17c}$$

$$=(ik+W_\infty)[S(k)/(k^2+B)](-ik+W_\infty)\,. \tag{17d}$$

These results apply equally well to single-channel scattering and to coupled channels with threshold energy differences where the diagonal matrix k must be replaced by the diagonal matrix of appropriate wave numbers and where we have already proven that W_∞ is diagonal.[12] Hence Eqs. (17) subsume the work of Refs. 2–5. Equations (17) are the central results of this work.

These results can also be obtained less formally by looking directly at the asymptotic form of the scattering wave function

$$\psi(k)=\phi e^{-ikr}-S(k)\phi e^{ikr}\,, \tag{18}$$

where ϕ is a constant column vector in the channel space representing the preparation or "controlling" of the incoming state. The supersymmetric partner scattering state is given asymptotically by

$$\begin{aligned}\tilde{\psi}(k)&=A_\infty^{-}\psi(k)\\&=(ik+W_\infty)\phi e^{-ikr}-(-ik+W_\infty)S(k)\phi e^{ikr}\,,\end{aligned} \tag{19}$$

where $A_\infty^{-}=-d/dr+W_\infty$. To read off $\tilde{S}$ we need to find the coefficient in the outgoing wave of the prepared state. We cannot simply divide through by $ik+W_\infty$ as if it were a normalization constant since it is, in fact, a matrix and the result of that division would not be an asymptotic state of $\tilde{H}$. Rather we write

$$\begin{aligned}\tilde{\psi}(k)=(ik+W_\infty)\phi e^{-ikr}&-(-ik+W_\infty)S(k)\\&\times(ik+W_\infty)^{-1}(ik+W_\infty)\phi e^{ikr}\,.\end{aligned} \tag{20}$$

From which we obtain $\tilde{S}$, the coefficient of the controlled state, in one of the equivalent forms of Eqs. (17).

It is trivial to see from (17) that $\tilde{S}$ is symmetric and unitary if S is also. The only unsettling feature of Eq. (17d) is that it appears to give $\tilde{S}$ a double pole at $E=-B$ (one from the explicit factor and one from the pole of S) while $\tilde{S}$ is supposed to have no pole at all. We shall prove that this double pole is exactly canceled by the two factors of $-ik+W_\infty$. For the one-channel problem where $W_\infty=-K$, $K^2=B$, it is clear that the factor of $(-ik+K)^2=-(k-iK)^2$ exactly cancels the double pole in the upper half plane leaving $\tilde{S}$ with no such pole. A corresponding proof follows in the case of coupled channels with threshold difference.

To proceed we need to calculate W_∞ in the case of no threshold differences. In that case we have for the asymptotic form of the Ψ of Eq. (5),

$$\Psi_\infty=(e^{-Kr}\varphi_1,e^{Kr}\varphi_2,\ldots,e^{Kr}\varphi_n)\,, \tag{21}$$

where the φ_i are constant (independent of r) column vectors. The constants in φ_1 are the asymptotic normalization constants of the bound state in the various channels. We can prove[13] that the φ_j $(j=2,\ldots,n)$ are orthogonal to φ_1. Since the $\varphi_2,\ldots,\varphi_n$ are linearly independent, they can be orthogonalized.

The fact that the φ_j's are orthogonal yields a very simple form for the Ψ_∞^{-1} from which we calculate W_∞ and obtain

$$\begin{aligned} W_\infty&=\Psi'_\infty\Psi_\infty^{-1}\\ &=K-2K(e^{-Kr}\varphi_1,0,\ldots,0)\Psi_\infty^{-1}\\ &=K(1-2P_1)\,, \end{aligned} \tag{22}$$

where P_1 is the projection operator onto the vector φ_1. This form for W_∞ is clearly Hermitian, and satisfies $W_\infty^2=B$. We can now write our S-matrix relation (17d) as

$$\tilde{S}(k)=-[k+iK(1-2P_1)][S(k)/(E+B)]\times[k+iK(1-2P_1)]\,, \tag{23}$$

which expresses $\tilde{S}$ in terms of S, the energy of the lowest state of H, $K^2=B$, and the asymptotic normalization constants of that state, which are in P_1. Thus the S-matrix relation (23) contains only physical asymptotic quantities.

To see that the pole of S at $E=-B$ is removed in $\tilde{S}$ we note that the residues of that pole of S in the various channels are just proportional to these same asymptotic normalization constants. Hence the pole part of S can be written as

$$S_{\text{pole}}=P_1/(E+B)\,. \tag{24}$$

The contribution of that pole in S to $\tilde{S}$ is found by substituting (24) in (23):

$$\tilde{S}_{\text{"pole"}}=-P_1(k-iK)^2/(E+B)^2 \tag{25}$$

which clearly has no pole in the upper half plane.

It may also seem strange that at high energy where one expects S to go over to unity, Eq. (23) gives $\tilde{S}=-1$. This arises from an extra centrifugal term in $\tilde{H}$ at small r that has the effect of raising the value of the orbital angular momentum quantum number, l, by one and giving a corresponding extra phase shift of $\pi/2$. This does not mean that S and $\tilde{S}$ correspond to different l; they do not, as the entire analysis given above was carried out after the partial-wave decomposition had been done.[14] It means rather that we have a singular potential in $\tilde{H}$.

We have shown that by combining supersymmetric quantum mechanics with the methods of formal scattering theory, one obtains simple relations between the S matrix and its supersymmetric partner involving only asymptotic quantities. For the single-channel case and for the case of coupled channels with threshold energy differences these relations were derived before use of less general methods. In the case with no threshold differences the relations between the S matrix and its supersymmetric partner involve not only the value of the binding energy of the lowest-energy bound state, removed by supersymmetry, but also the asymptotic normalization constants (or S-matrix pole residues) of that state. Our results are of interest as an extension of SSQM to coupled-channel scattering. They may also have interesting applications as they can be iterated and inverted to give expressions for S in terms of an $\tilde{S}$ of a possibly simpler problem. For example, $\tilde{S}$ may be diagonal while S is not. We are examining an application to the coupled $l=0$ and $l=2$ waves of the deuteron.

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[1]E. Witten, Nucl. Phys. **B188**, 513 (1985).

[2]R. D. Amado, Phys. Rev. A **37**, 2277 (1988).

[3]D. Baye, Phys. Rev. Lett. **58**, 2738 (1987).

[4]F. Cooper, J. N. Ginocchio, and A. Wipf, Phys. Lett. A **129**, 145 (1988).

[5]R. D. Amado, F. Cannata, and J. P. Dedonder, Phys. Rev. A **38**, 3797 (1988).

[6]Factorization is an old idea dating from the work of G. Darboux in 1882. For an excellent pedagogic review see Marshall Luban and D. L. Pursey, Phys. Rev. D **33**, 431 (1986); D. L. Pursey, Phys. Rev. D **33**, 1048 (1986), and **36**, 1103 (1987).

[7]Also of pedagogic interest is V. Sukumar, J. Phys. A **18**, 2917, 2937 (1985), and **19**, 2297 (1986), and **20**, 2461 (1987), and **21**, L455 (1988).

[8]W. Kwong and J. L. Rosner, Prog. Theor. Phys. Suppl. **86**, 366 (1986); A. Khare and U. P. Sukhatme, J. Phys. A **21**, L501 (1988); L. E. Gendenshtein, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 299 (1983) [JETP Lett. **38**, 356 (1983)].

[9]F. Cooper, J. N. Ginocchio, and A. Khare, Phys. Rev. D **36**,

2458 (1987).

[10]From $H\Psi=-B\Psi$ and its Hermitian conjugate one finds the Wronskian matrix relation ("current conservation"): $(d/dr)[\Psi^\dagger(d\Psi/dr)-(d\Psi^\dagger/dr)\Psi]=0$. From the regularity at the origin one further obtains $\Psi^\dagger(d\Psi/dr)-(d\Psi^\dagger/dr)\Psi=0$. The Hermiticity of W follows from this last relation since $W-W^\dagger=\Psi'\Psi^{-1}-\Psi^{\dagger -1}\Psi^{\dagger\prime}=\Psi^{\dagger -1}(\Psi^\dagger\Psi'-\Psi'^\dagger\Psi)\Psi^{-1}=0$.

[11]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964); R. G. Newton, *Scattering Theory of Waves and Particles* (Springer-Verlag, New York, 1982), p. 156ff; J. R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions* (Wiley, New York, 1972).

[12]Similar results for the one-channel problem using the factorization method have been obtained by a number of authors. In particular Faddeev [L. D. Faddeev, J. Math. Phys. **4**, 72 (1963)] reviews detailed results of Krein [M. G. Krein, Dokl. Akad. Nauk SSSR **113**, 970 (1957)].

[13]The component $1,j$ of the Wronskin matrix relation given in Ref. 10 is $\psi_1'^\dagger\psi_j-\psi_1^\dagger\psi'=0$. Inserting the asymptotic form (21), one obtains $-2K\varphi_1^\dagger\varphi_j=0$ which proves the orthogonality of φ_1 to all φ_j's, $j=2,\ldots,n$.

[14]Other authors [e.g., Krein and Faddeev (Ref. 12) and Sukumar (Ref. 7)] interpret these results as corresponding to shifted angular momentum. It would indeed be interesting to obtain a consistent SSQM in three dimensions in which such an interpretation was manifest.