# Rotational Anomaly and Fractional Spin in the Gauged $CP^1$ Nonlinear $\sigma$ Model with the Chern-Simons Term

Prasanta K. Panigrahi, Shibaji Roy, and Wolfgang Scherer
*Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627*
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We consider the interacting $CP^1$ nonlinear $\sigma$ model in $2+1$ dimensions in the presence of an Abelian Chern-Simons term and quantize it canonically using the Dirac-Bergmann algorithm. There arises an anomalous term in the angular momentum in presence of the interaction, which in the long-range limit yields the fractional spin for the excitations.

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It is known that $(2+1)$-dimensional field theories can have excitations with fractional spin and statistics, the often quoted example being the O(3) nonlinear $\sigma$ model with the Hopf term in the action.[1] In this model the configuration space of the finite-energy soliton solution is $C=\{n\,|\,n{:}s^2\to s^2\}$. Since $\pi_1(C)=\pi_3(s^2)=Z$, the configuration space is infinitely connected and hence admits the possibility of fractional spin and statistics, which have been studied quite exhaustively in the literature.[2]

The $CP^1$ model which is intimately related to the O(3) nonlinear $\sigma$ model[3] in the long-range limit has received a lot of attention recently in connection with high-$T_c$ superconductivity.[4] In Ref. 4 Polyakov analyzed the interacting $CP^1$ model in the presence of a Chern-Simons (CS) term and asserted in a path integral formulation that the short-distance bosonic excitations of the theory become fermions at long distance. The CS term, which is nothing but the Hopf invariant in disguise, has been shown to be induced in a fermionic theory coupled with scalar fields of the nonlinear $\sigma$ model.[5] $(2+1)$-dimensional gauge theories with the CS term have been studied extensively in the literature.[6] The presence of the CS term in the action is crucial in converting the dressed $z$ quanta into fermions and has deep connection with topological invariants associated with a given space curve.[7] An interacting fermionic field theory has been considered earlier in the presence of an Abelian CS term by Hagen.[8] The canonical quantization of the above model yields an anomalous term in the angular momentum, the presence of the CS term being essential to the appearance of this rotational anomaly.

The purpose of this Letter is to quantize canonically the interacting nonlinear $\sigma$ model considered by Polyakov, with use of the method of Dirac brackets, and show the connection between the rotational anomaly and fractional spin in the theory. The consideration of the relevant current algebra and other details of the calculation will be provided elsewhere.

The Lagrangian of the model we consider is

$$L=\left(\frac{1}{\gamma_0}\overline{D_\mu z_k}D^\mu z_k+\frac{\theta}{4\pi^2}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda\right). \tag{1}$$

Here $k=1,2$ and $z_k$ is a two-component complex field which satisfies the constraint

$$z_k z_k^*=|z_1|^2+|z_2|^2=1,$$

and the covariant derivative $D_\mu$ is $\partial_\mu-iA_\mu$, where $A_\mu$ is the gauge field. The Levi-Civita symbol $\epsilon_{\mu\nu\lambda}$ is fixed by $\epsilon_{012}=1$ and $g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$. The equation of motion for the $A_\mu$ field is given as

$$\frac{1}{\gamma_0}J_\mu+\frac{\theta}{2\pi^2}\epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda=0, \tag{2}$$

where

$$J_\mu=i(z_k^*D_\mu z_k-\overline{D_\mu z_k}z_k). \tag{3}$$

The equation of motion implies the conservation law

$$\partial_\mu J^\mu=0. \tag{4}$$

In the Coulomb gauge, i.e., $\partial^i A_i=0$, the equation of motion for the $A_\mu$ field yields two constraints, which can be easily demonstrated by considering the temporal and spatial components of Eq. (2) separately. They are

$$\frac{1}{\gamma_0}J_0+\frac{\theta}{2\pi^2}\epsilon_{ij}\partial^i A^j=0, \tag{5}$$

where

$$\epsilon_{ij}=\epsilon_{0ij}\ \text{and}\ i,j=1,2$$

and

$$\frac{1}{\gamma_0}J_i+\frac{\theta}{2\pi^2}\epsilon_{ij}(\partial^j A^0-\partial^0 A^j)=0$$

or

$$\epsilon_{ik}\,\partial_k J_i=\frac{\gamma_0\theta}{2\pi^2}\nabla^2 A_0. \tag{6}$$

These equations do not involve explicit time dependence and hence are constraints. In the Coulomb gauge $A^i$ can be written as

$$A^i(x)=\epsilon^{ij}\partial_j\Psi(x). \tag{7}$$

Equation (5) after substitution of the above expression

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for $A^i(x)$ gives

$$J_0(x) = -\frac{\gamma_0\theta}{2\pi^2}\nabla^2\Psi(x).$$

Hence

$$\Psi(x) = \frac{2\pi^2}{\gamma_0\theta}\int d^2x' D(x-x')J_0(x'), \tag{8}$$

where the Green's function $D(x-x')$ satisfies

$$\nabla^2 D(x-x') = -\delta(x-x')$$

and so

$$A^i(x) = \frac{2\pi^2}{\gamma_0\theta}\epsilon^{ij}\partial_j\int d^2x' D(x-x')J_0(x'). \tag{9}$$

From Eq. (6) we get, after an integration by parts,

$$A_0(x) = \frac{2\pi^2}{\gamma_0\theta}\epsilon^{ik}\int d^2x' J_i(x')\partial_k' D(x-x'). \tag{10}$$

In two space dimensions

$$D(x-x') = -\frac{1}{4\pi}\ln|x-x'|^2 + \text{const}.$$

Such results for the radiation-gauge potentials have been derived in Ref. 8 using the current of a fermionic field.

We now proceed to quantize canonically the theory using the Dirac-Bergmann algorithm.[9] The whole set of constraints is given below.

$$\begin{aligned}
&\phi_1 = z_k^* z_k - 1 \approx 0, \quad \phi_2 = \pi_k z_k + \pi_k^* z_k^* \approx 0, \quad \phi_3 = \pi_k z_k - \pi_k^* z_k^* + \frac{i\theta}{2\pi^2}\epsilon_{ij}\partial^i A^j \approx 0, \quad \phi_4 = P_0 \approx 0,\\
&\phi_5 = P_1 - \frac{\theta}{4\pi^2}A^2 \approx 0, \quad \phi_6 = P_2 + \frac{\theta}{4\pi^2}A^1 \approx 0, \quad \phi_7 = \partial_i A^i \approx 0,\\
&\phi_8 = \nabla^2 A_0 + \frac{4\pi^2}{\gamma_0\theta}(\partial_1 z_k\,\partial_2 z_k^* - \partial_2 z_k^*\,\partial_1 z_k^* + i\,\partial_1 A_2 - i\,\partial_2 A_1) \approx 0,
\end{aligned} \tag{11}$$

where $\pi_k$, $\pi_k^*$ are the momenta conjugate to $z_k$ and $z_k^*$, respectively. $P_0$, $P_1$, and $P_2$ are the momenta, conjugate to $A_0$, $A_1$, and $A_2$ in that order. All the above constraints are second class and the restricted phase-space dimension is $14-8=6$. The nonvanishing set of equal-time Dirac brackets is

$$\begin{aligned}
&\{z_k(x),\pi_l(y)\}_D = [\delta_{kl} - \tfrac{1}{2}z_k(x)z_l^*(x)]\delta(x-y), \quad \{\pi_k(x),\pi_l(y)\}_D = \tfrac{1}{2}[\pi_k(x)z_l^*(y) - z_k^*(x)\pi_l(y)]\delta(x-y),\\
&\{\pi_k(x),\pi_l^*(y)\}_D = \tfrac{1}{2}[\pi_k(x)z_l(y) - z_k^*(x)\pi_l^*(y)]\delta(x-y), \quad \{z_k^*(x),\pi_l(y)\}_D = -\tfrac{1}{2}z_k^*(x)z_l^*(x)\delta(x-y),\\
&\{A_i(x),z_k(y)\}_D = \frac{2\pi^2 i}{\theta}\frac{1}{\nabla_x^2}\epsilon_{ij}\,\partial_j^x\delta(x-y)z_k(y), \quad \{A_i(x),\pi_k(y)\}_D = -\frac{2\pi^2 i}{\theta}\frac{1}{\nabla_x^2}\epsilon_{ij}\,\partial_j^x\delta(x-y)\pi_k(y),\\
&\{P_i(x),z_k(y)\}_D = i\frac{1}{2\nabla_x^2}\partial_i^x\delta(x-y)z_k(y), \quad \{P_i(x),\pi_k(y)\}_D = -i\frac{1}{2\nabla_x^2}\partial_i^x\delta(x-y)\pi_k(y).
\end{aligned} \tag{12}$$

We then quantize the system by replacing $\{\,,\}_D$ by $-i\,[\,,\,]$.

After achieving the quantization we proceed to compute the angular momentum in the given model. The angular momentum operator in $2+1$ dimensions is

$$J = \tfrac{1}{2}\epsilon_{ij}J^{ij} = \tfrac{1}{2}\epsilon_{ij}[J^{0i},J^{0j}]. \tag{13}$$

In terms of the energy-momentum tensor

$$J = \epsilon_{ij}\int d^2x\, x^i T^{0j}.$$

The symmetric energy-momentum tensor can be obtained by coupling the fields to gravity and then varying the action with respect to $g^{\mu\nu}$:

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{\gamma_0}(\overline{D_\mu z_k}D_\nu z_k + \overline{D_\nu z_k}D_\mu z_k) - \frac{g_{\mu\nu}}{\gamma_0}\overline{D_\rho z_k}D^\rho z_k. \tag{14}$$

Hence

$$T_{0j} = \frac{1}{\gamma_0}(\overline{D_0 z_k}D_j z_k + \overline{D_j z_k}D_0 z_k).$$

The canonical momenta conjugate to $z_k$ and $z_k^*$ are

$$\pi_k = \frac{1}{\gamma_0}\overline{D_0 z_k} \text{ and } \pi_k^* = \frac{1}{\gamma_0} D_0 z_k ,$$

respectively.

To find out the rotational property of the $z_k$ fields we compute the commutator

$$\begin{aligned}[J, z_k(y)] &= \epsilon_{ij}\int d^2x\, x^i [T^{0j}(x), z_k(y)] \\ &= \epsilon_{ij}\int d^2x\, x^i [\pi_l\, \partial^j z_l(x) + \partial^j z_l^* \pi_l^*(x) - i\pi_l A^j z_l(x) + iA^j z_l^* \pi_l^*(x), z_k(y)] \\ &= \epsilon_{ij}\int d^2x\, x^i [\pi_l\, \partial^j z_l(x) + \partial^j z_l^* \pi_l^*(x), z_k(y)] - i\epsilon_{ij}\int d^2x\, x^i [\pi_l A^j z_l(x) - A^j z_l^* \pi_l^*(x), z_k(y)] \\ &= i(\mathbf{y}\times\nabla) z_k(y) + \frac{\pi}{\gamma_0 \theta} Q z_k(y) ,\end{aligned} \tag{15}$$

where $Q = \int J_0(x)\, d^2x$ is the electromagnetic charge operator. The reason for the appearance of the anomaly term can be clarified if we trace the anomalous term from the angular momentum directly.

In

$$J = \epsilon_{ij}\int d^2x\, x^i T^{0j} ,$$

we concentrate on the term giving rise to the rotational anomaly, which can be identified as

$$J_{\text{anomalous}} = \epsilon_{ij}\int d^2x\, x^i [-iA^j(\pi_k z_k - \pi_k^* z_k^*)] .$$

Making use of Eq. (11) this can be written as

$$\begin{aligned}J_{\text{anomalous}} &= -\frac{\theta}{2\pi^2}\epsilon_{ij}\epsilon_{kl}\int d^2x\, x^i A^j \partial^k A^l \\ &= \frac{\pi Q^2}{2\gamma_0^2 \theta} .\end{aligned} \tag{16}$$

The final answer has been obtained after plugging in the expression of the $A_\mu$ field in terms of the current of the theory. An anomaly in the angular momentum which is proportional to the square of the charge operator is known[8] to characterize the fermionic case as well. The anomalous term shows that the angular momentum of the excitations of the system contains, in addition to the regular orbital part, an anomalous part which contributes as fractional spin to the system. This gives credence to Polyakov's assertion that the fermions transmute to a spin-zero and a spin-one object in the presence of a CS term.

The long-distance (as $|x| \to \infty$) or small-momentum behavior of the model can be understood through the finite-energy solution of $A_\mu$ as $-iz_k^* \partial_\mu z_k$. At the space-time asymptote the CS term becomes a Hopf invariant for the map $z: s^3 \to CP^1 \approx s^2$. So, the gauge field in the interacting theory becomes asymptotically topological and one can define a conserved topological current as

$$J_\mu^t = -\frac{i}{2\pi}\epsilon_{\mu\nu\lambda}\partial^\nu z_k^* \partial^\lambda z_k . \tag{17}$$

Equation (5), therefore, gives a connection between the electromagnetic charge and the topological charge operator, the relation being

$$Q = \frac{\gamma_0 \theta}{\pi} Q^t , \tag{18}$$

where

$$Q^t = \int d^2x\, J_0^t(x) .$$

It is worth mentioning at this point that the normalization for the topological current in the text yields $Q^t = 1$ for the one-soliton sector of the theory.

Making use of Eq. (18) the angular momentum of the system becomes

$$J = \epsilon_{ij}\int d^2x\, x^i (\pi_k\, \partial^j z_k + \partial^j z_k^* \pi_k^*) + \frac{\theta}{2\pi} Q^{t^2} . \tag{19}$$

At $\theta = \pi$ and for the one-soliton sector, the contribution of the rotational anomaly is one-half which precisely agrees with the result of Wilczek and Zee.[1]

In conclusion, we would like to point out that although the result obtained in this Letter is expected in light of the long-distance equality of the $CP^1$ model with the O(3) nonlinear $\sigma$ model, several interesting points emerge from the analysis. The identification of the rotational anomaly in the $CP^1$ model with the fractional spin in the O(3) $\sigma$ model follows from the long-distance equivalence of the electromagnetic current of the former model with the topological current of the later. This feature strongly resembles two-dimensional field theories where the electromagnetic current of a fermionic theory is identified with the topological current of its bosonic counterpart. This will make it easier to analyze the spectrum of the theory in terms of the $z$ fields where the excitations are point particles in contrast to the extended nature of the solitonic excitations of the O(3) $\sigma$ model. Furthermore, the absence of radiative corrections to the CS term in this model makes the tree-level result exact.

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